A 2-parametric generalization of Sierpiński gasket graphs

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Abstract

Graphs $S[n,k]$ are introduced as the graphs obtained from the Sierpiński graphs $S(n,k)$ by contracting edges that lie in no complete subgraph K_k. The family $S[n,k]$ is a generalization of a previously studied class of Sierpiński gasket graphs S_n. Several properties of graphs $S[n,k]$ are studied in particular, hamiltonicity and chromatic number.

Key words: Sierpiński graphs; Sierpiński gasket graphs; Hamiltonicity; Chromatic number

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1 Introduction

Sierpiński-like graphs appear in many different areas of graph theory, topology, probability ([8, 11]), psychology ([16]), etc. The special case $S(n,3)$ turns out to be only a step away of the famous Sierpiński gasket graphs S_n—the graphs obtained after a finite number of iterations that in the limit give the Sierpiński gasket, see [10]. This connection was introduced by Grundy, Scorer and Smith in [23] and later observed in [24].

Graphs $S(n,3)$ are also important for the Tower of Hanoi game since they are isomorphic to Hanoi graphs with n discs and 3 pegs. Metric properties, planarity, vertex and edge coloring were studied by now, see, for instance [1, 5, 6, 7, 14, 22]. Furthermore, in [12] it is proved that graphs $S(n,3)$ are uniquely 3-edge-colorable and have unique Hamiltonian cycles.

Graphs $S(n,3)$ can be generalized to Sierpiński graphs $S(n,k)$, $k \geq 3$, which are also called *Klavžar-Milutinović graphs* and denoted KM_{nk}

[17]. The motivation came from topological studies of the Lipscomb's space [18, 19]. Graphs $S(n,k)$ independently appeared in [21]. These graphs have many interesting properties, for instance, coding [4] and metric properties [20]. Moreover, in [13] it is shown that graphs $S(n,k)$ are Hamiltonian and that there are at most two shortest paths between any pair of their vertices. The length between any two vertices can be determined in $O(n)$ time.

Sierpiński graphs are almost regular. In a natural way, two new families of regular Sierpiński-like graphs $S^{+}(n,k)$ and $S^{++}(n,k)$ were introduced in [15] and their crossing numbers determined (in terms of the crossing number of complete graphs).

Since Sierpiński gasket graphs S_n are important and are naturally derived from Sierpiński graphs $S(n,3)$, we can apply the same construction for $S(n,k)$, where $k \geq 3$. We do this in section 2 and denote the new graphs $S[n,k]$. In [26] one can find a more algorithmic definition of S_n based on trilinear coordinates, whereas in [3] one can find the definition of the Sierpiński gasket SG_d in any Euclidean space of dimension d with the number of vertices and number of edges. Their graphs $SG_{d+1}(n)$ are the generalized Sierpiński gasket graphs $S[n,k]$ introduced in this paper.

It is well known that graphs $S(n,k)$, $k \geq 3$, are Hamiltonian [13], as are graphs S_n [25]. We shall prove the same result for graphs $S[n,k]$. The chromatic number, chromatic index and total chromatic number of the Sierpiński graphs $S(n,k)$ were already determined [20, 9]. In [9], a question was posted for the total chromatic number, where k is even. The question was answered in [7]. The same chromatic properties were studied for the Sierpiński gasket graphs. The chromatic number was determined in [25], chromatic index in [12] and total chromatic number in [9]. In this paper, we study the chromatic number of graphs $S[n,k]$.

2 Graphs $S[n,k]$ and their basic properties

First we recall the definition of the Sierpiński graphs $S(n,k)$. They are defined for $n \geq 1$ and $k \geq 1$ as follows. The vertex set of $S(n,k)$ consists of all n-tuples of integers $1, 2, \ldots, k$, that is, $V(S(n,k)) = \{1, 2, \ldots, k\}^n$. Two different vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are adjacent if and only if there exists an $h \in \{1, \ldots, n\}$ such that

(i) $u_t = v_t$, for $t = 1, \ldots, h-1$;

(ii) $u_h \neq v_h$; and

(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h+1, \ldots, n$.

We will shortly write $u_1u_2 \ldots u_n$ for $(u_1, u_2, \ldots, u_n)$. See Fig. 1 for $S(2,4)$.

By fixing $u_1 \in \{1, \ldots, k\}$, we get $S(n-1,k)$. In other words, $S(n,k)$ is constructed of k different $S(n-1,k)$. We label each with $S_i(n,k)$, for every $i \in \{1, \ldots, k\}$. Note that $S_i(n,k)$ and $S_j(n,k)$, $i \neq j$, are connected with a single edge between vertices $ij \dots j$ and $ji \dots i$. As in [9] we call these edges the linking edges of $S(n,k)$.

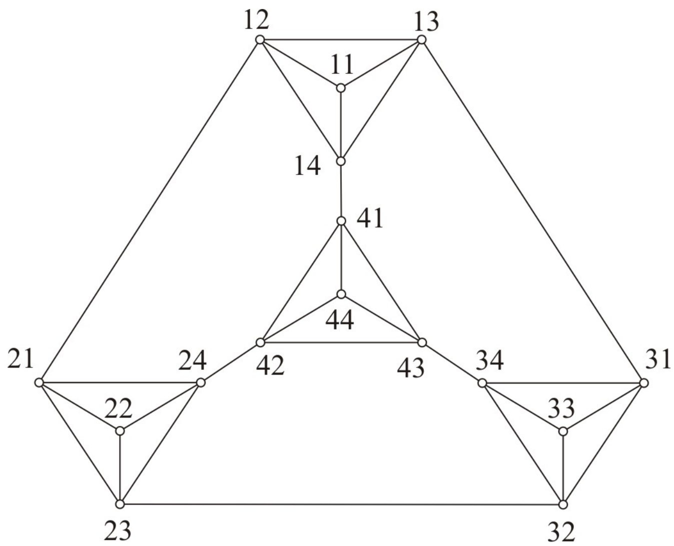

Figure 1: Graph $S(2,4)$

In [12] in a natural way the Sierpiński gasket S_n is constructed from the Sierpiński graph $S(n,3)$ by contracting all of its edges that lie in no triangle K_3. We can apply the same construction method for any Sierpiński graph $S(n,k)$ by contracting all of its edges that lie in no induced subgraph K_k.

Let $u_1u_2 \dots u_r jl \dots l$ and $u_1u_2 \dots u_r lj \dots j$, $0 \le r \le n-2$, be two adjacent vertices of graph $S(n,k)$. We identify them in one vertex and write $u_1u_2 \dots u_r\{j,l\}$ or shortly $u_{(r)}\{j,l\}$, $j \neq l$ and $j,l \in \{1,\dots,k\}$, where $u_{(0)}\{j,l\}$ means $\{j,l\}$. We get a 2-parametric Sierpiński gasket graph $S[n,k]$ (shortly k-Sierpiński gasket graph). We already know the case for $k=3$, namely the Sierpiński gasket $S_n = S[n,3]$. An example of $S[2,4]$ is shown in the Fig. 2.

Since $S(n,k)$ is built of k copies of $S(n-1,k)$, the graph $S[n,k]$ is also built of k copies of $S[n-1,k]$. We denote each copy with $S_i[n,k]$. Note that $S_i[n,k]$ and $S_j[n,k]$, $i \neq j$, share one vertex, that is $\{i,j\}$.

It is known, that $S(1,k)$ is isomorphic to K_k. Hence, $S[1,k]$ is also isomorphic to K_k.

Adjacency of vertices in $S[n,k]$ is given in the next proposition.

Proposition 2.1 *Let $n \ge 2$ and $u = u_1 \dots u_r\{i,j\}$ be a vertex in $S[n,k]$, $i,j \in \{1,\dots,k\}, i \neq j$.*

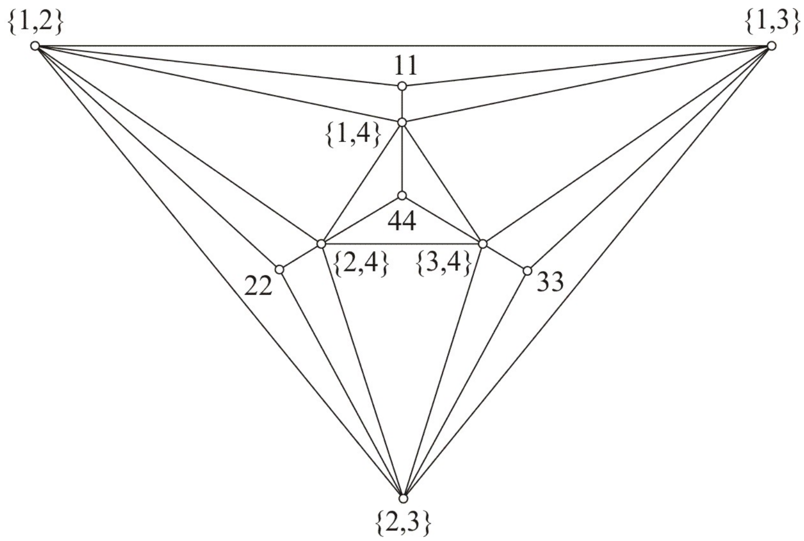

Figure 2: Graph $S[2,4]$

(i) *If $0 \leq r \leq n-3$, then u is adjacent to*

$$\overline{u}_{(n-2)}\{j,l\}, l \in \{1,\ldots,k\}\backslash\{j\},$$

and

$$\overline{\overline{u}}_{(n-2)}\{i,l\}, l \in \{1,\ldots,k\}\backslash\{i\}.$$

(ii) *If $r = n-2$, then u is adjacent to*

$$\overline{u}_{(n-2)}\{i,l\}, l \in \{1,\ldots,k\}\backslash\{i,j\},$$

$$\overline{\overline{u}}_{(n-2)}\{j,l\}, l \in \{1,\ldots,k\}\backslash\{i,j\},$$

$$\begin{cases} \overline{u}_{(t-1)}\{i,u_t\}, & t \text{ largest index with } u_t \neq i,\ 1 \leq t \leq n-2, \\ i\ldots i, & \text{else}, \end{cases}$$

and

$$\begin{cases} \overline{\overline{u}}_{(s-1)}\{j,u_s\}, & s \text{ largest index with } u_s \neq j,\ 1 \leq s \leq n-2, \\ j\ldots j, & \text{else}. \end{cases}$$

Proof. *(i)* Let $r \leq n-3$. Take two vertices $\overline{u} = u_1 \dots u_r ij \dots jj$ and $\overline{\overline{u}} = u_1 \dots u_r ji \dots ii$, where $\overline{u}$ ends with at least two j's and $\overline{\overline{u}}$ with at least two i's. Then $\overline{u}$ is in $S(n,k)$ adjacent to $u_1 \dots u_r ij \dots jl$, $l \in \{1,\dots,k\} \backslash \{j\}$. In the construction procedure this vertex contracts to $\overline{u}_{(n-2)}\{j,l\}$, $l \in \{1,\dots,k\} \backslash \{j\}$ in $S[n,k]$. Similarly, $\overline{\overline{u}}$ is adjacent to $u_1 \dots u_r ji \dots il$, $l \in \{1,\dots,k\} \backslash \{i\}$, which contracts to $\overline{\overline{u}}_{(n-2)}\{i,l\}$, $l \in \{1,\dots,k\} \backslash \{i\}$ in $S[n,k]$. The argument also holds for $r = 0$.

(ii) Let $r = n-2$. Then $\overline{u} = u_1 \dots u_{n-2} ij$ and $\overline{\overline{u}} = u_1 \dots u_{n-2} ji$. In $S(n,k)$, the vertex $\overline{u}$ is adjacent to $u_1 \dots u_{n-2} il$, $l \in \{1,\dots,k\} \backslash \{i,j\}$, which contracts to $\overline{u}_{(n-2)}\{i,l\}$, $l \in \{1,\dots,k\} \backslash \{i,j\}$ in $S[n,k]$. It is also adjacent to $x = u_1 \dots u_{n-2} ii$. If $u_1 = \dots = u_{n-2} = i$, then $\overline{u}$ is adjacent to the extreme vertex $i \dots i$ in $S[n,k]$. If not all of $u_1,\dots,u_{n-2}$ are equal to i, then let t be the largest index that $u_t \neq i$. Then the vertex $\overline{u}$ is adjacent to $x = u_1 \dots u_t i \dots i$, $t \in \{1,\dots,n-2\}$. In this case, $\overline{u}$ is adjacent to $\overline{u}_{(t-1)}\{i,u_t\}$. Similarly, the vertex $\overline{\overline{u}}$ is adjacent to $u_1 \dots u_{n-2} jl$, $l \in \{1,\dots,k\} \backslash \{i,j\}$, which contracts to $\overline{\overline{u}}_{(n-2)}\{i,l\}$, $l \in \{1,\dots,k\} \backslash \{i,j\}$ in $S[n,k]$. The vertex $\overline{\overline{u}}$ is also adjacent to $x = u_1 \dots u_{n-2} jj$. If $u_1 = \dots = u_{n-2} = j$, then $\overline{\overline{u}}$ is adjacent to the extreme vertex $j \dots j$ in $S[n,k]$. If not all of $u_1,\dots,u_{n-2}$ are equal to j, then let s be the largest index that $u_s \neq j$. Then the vertex $\overline{\overline{u}}$ is adjacent to $y = u_1 \dots u_s j \dots j$, $s \in \{1,\dots,n-2\}$. Therefore, $\overline{\overline{u}}$ is adjacent to $\overline{\overline{u}}_{(t-1)}\{i,u_t\}$. □

It is easy to see that the graphs $S[n,k]$ have k extreme vertices of degree $k-1$ and $|V(S[n,k])| - k$ remaining vertices of degree $(k-1)+(k-1) = 2(k-1)$. We immediately get:

Proposition 2.2 *Graphs $S[n,k]$ are Eulerian if and only if k is odd.*

To gain more knowledge of the structure of $S[n,k]$, we give the number of vertices and edges of $S[n,k]$.

Proposition 2.3 *Graphs $S[n,k]$ have $\frac{k}{2}\left(k^{n-1}+1\right)$ vertices and $\frac{k-1}{2} \cdot k^n$ edges.*

Proof. A vertex in $S[n,k]$ is of the form $u_1 u_2 \dots u_r \{j,l\}$. We have k possibilities for every u_i, $i \in \{1,\dots,r\}$, and $\binom{k}{2}$ possibilities for the unordered pair $\{j,l\}$. There are also k extreme vertices left. We get:

$$|V(S[n,k])| = k + \sum_{r=0}^{n-2} k^r \cdot \binom{k}{2} = \frac{k}{2}(k^{n-1}+1).$$

We determine the number of edges by using the formula:

$$\begin{aligned}
|E(S[n,k])| &= \frac{1}{2}\sum_{u\in S[n,k]} deg(u) \\
&= \frac{1}{2}\left(k(k-1) + \left(\frac{k}{2}\left(k^{n-1}+1\right)-k\right)\cdot 2(k-1)\right) \\
&= \binom{k}{2}k^{n-1} = \frac{k-1}{2}\cdot k^n.
\end{aligned}$$

□

Remark 2.4 *Alternatively, for all $k \geq 1$, a proof can be based on the recurrence relations $|V(S[n+1,k])| = k\cdot|V(S[n,k])| - \binom{k}{2}$, $n \geq 1$, $|V(S[1,k])| = k$, since $\binom{k}{2}$ vertices are merged and $S[1,k]$ is isomorphic to K_k. Further, $|E(S[n+1,k])| = k\cdot|E(S[n,k])|$, $n \geq 1$, $|E(S[1,k])| = \binom{k}{2}$, since the graph $S[n+1,k]$ consists of k copies of $S[n,k]$.*

Corollary 2.5 *For all $n,k \in \mathbb{N}$: $|E(S[n,k])| = (k-1)\cdot|V(S[n,k])| - \binom{k}{2}$.*

Applying Proposition 2.3 and setting $k = 3$ we get:

Corollary 2.6 *Graphs S_n have $\frac{3}{2}(3^{n-1}+1)$ vertices and 3^n edges.*

3 Hamiltonicity

In [25], Tequia and Godbole proved that graphs S_n are Hamiltonian. In this section we generalize their statement to $S[n,k]$. First, we need a lemma.

Lemma 3.1 *Graphs $S[n,k]$, $k \geq 2$, have a Hamiltonian path connecting two arbitrary extreme vertices.*

Proof. The statement holds for $k = 2$ because $S[n,2]$ is isomorphic to a path on $2^{n-1}+1$ vertices.

Let $k \geq 3$. This statement is true for $n = 1$, since $S[1,k]$ is isomorphic to K_k. Let $n \geq 2$. Without lose of generality we start in vertex $1\ldots1$. Since $S_1[n,k]$ is isomorphic to $S[n-1,k]$ we can find a Hamiltonian path from $1\ldots1$ to the vertex $\{1,2\}$. With the same argument, we can find a Hamiltonian path in $S_2[n,k]$ from the vertex $\{1,2\}$ to the vertex $\{2,3\}$. Next we find a Hamiltonian path in $S_3[n,k]$ from the vertex $\{2,3\}$ to the vertex {3,4} by avoiding {1,3} (since we locally have an induced complete graph, avoiding is possible). We continue the procedure in $S_4[n,k]$ by finding a Hamiltonian path between vertices $\{3,4\}$ and $\{4,5\}$ by avoiding vertices $\{1,4\}$ and $\{2,4\}$. In general we find a Hamiltonian path in $S_i[n,k]$, $i \in \{3,\ldots,k-1\}$, from the vertex $\{i-1,i\}$ to the vertex $\{i,i+1\}$ by avoiding vertices $\{1,i\},\{2,i\},\ldots,\{i-2,i\}$. Finally, we find a Hamiltonian path in $S_k[n,k]$ from vertex $\{k-1,k\}$ to vertex $k\ldots k$ by avoiding vertices $\{1,k\},\{2,k\},\ldots,\{k-2,k\}$. All together we have constructed a Hamiltonian path between vertices $1\ldots1$ and $k\ldots k$.

Similarly, we can find a Hamiltonian path between any two different extreme vertices in $S[n,k]$. □

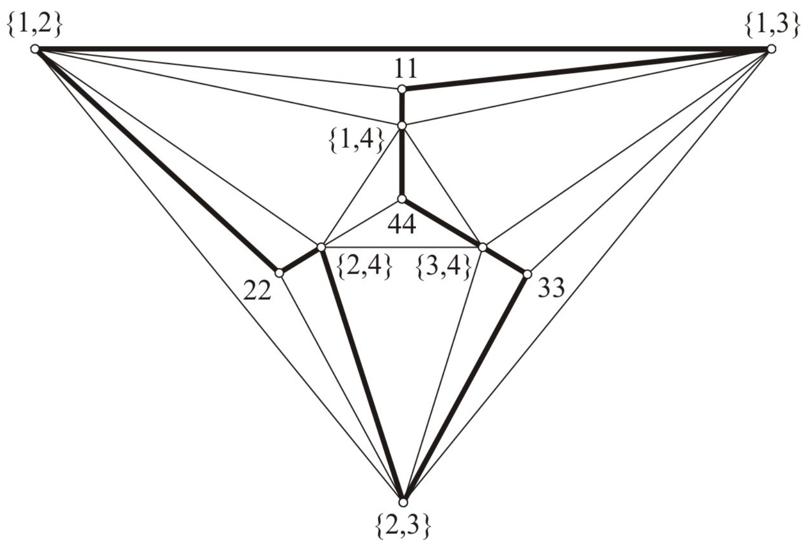

Figure 3: A Hamiltonian cycle in $S[2,4]$

Theorem 3.2 *Graphs $S[n,k]$ are Hamiltonian, for any $n \geq 1$ and $k \geq 3$.*

Proof. The statement is true for $n=1$, since $S[1,k]$ is a complete graph.

Let $n \geq 2$. By Lemma 3.1, we can find a Hamiltonian path from the vertex $\{k,1\}$ to the vertex $\{1,2\}$ in $S_1[n,k]$. Similarly, we can find a Hamiltonian path between vertices $\{1,2\}$ and $\{2,3\}$ in $S_2[n,k]$. Now we find a Hamiltonian path in $S_3[n,k]$ from between vertices $\{2,3\}$ and $\{3,4\}$ by avoiding vertex $\{1,3\}$. In general, we find a Hamiltonian path in $S_i[n,k]$ between vertices $\{i-1,i\}$ and $\{i,i+1\}$, $i \in \{3,\ldots,k-1\}$, by avoiding vertices $\{1,i\},\{2,i\},\ldots,\{i-2,i\}$. Finally, we find a Hamiltonian path in $S_k[n,k]$ from the vertex $\{k-1,k\}$ to the vertex $\{k,1\}$ by avoiding vertices $\{2,k\},\ldots,\{k-2,k\}$. Again, all the avoiding is possible because locally we have an induced complete graph. All the paths together form a Hamiltonian cycle in $S[n,k]$. $\square$

Fig. 3 shows a Hamiltonian cycle in $S[2,4]$ obtained with the above construction. Once again, by setting $k=3$ we get:

Corollary 3.3 [25] *Graphs S_n are Hamiltonian.*

4 The chromatic number

Since $S[n,k]$ is built of complete graphs K_k, it is obvious that $\chi(S[n,k]) \geq k$. We prove the following.

Theorem 4.1 *For any $n \geq 1$ and any $k \geq 1$, $\chi(S[n,k]) = k$.*

Proof. For $k=1$ and $k=2$ we get a vertex and a path on $2^{n-1}+1$ vertices respectively for which the statement is true. Let $k \geq 3$. For $n=1$ we have a complete graph on k vertices. It is well known that $\chi(K_k) = k$. Let $n \geq 2$.

Graph $S[n,k]$ consists of k copies of $S[n-1,k]$, each denoted with $S_i[n,k]$, $i \in \{1,\ldots,k\}$. Two copies, say $S_i[n,k]$ and $S_j[n,k]$, $i \neq j$, share a common vertex $\{i,j\}$. For every $i,j \in \{1,\ldots,k\}$, $i \neq j$, expand this vertex into two vertices $ij\ldots j$ and $ji\ldots i$ connected with a linking edge.

Note that for $n \geq 3$ the expansion process described above is not the inverse process of getting $S[n,k]$ from the $S(n,k)$, since we do not expand all the vertices that were contracted in $S(n,k)$.

By induction assumption graph $S_1[n,k]$ can be colored with k colors. Denote the color of the vertex $1j\ldots j$, $j \in \{1,\ldots,k\}$, with $c_{1+j-2 \pmod k}$, where $c_{1+j-2 \pmod k} \in \{1,\ldots,k\}$. Then graphs $S_i[n,k]$, $i \in \{2,\ldots,k\}$, can also be colored in such a way that a vertex $ij\ldots j$ receives color $c_{i+j-2 \pmod k} \in \{1,\ldots,k\}$. This coloring is the same as the coloring used to color graph $S_1[n,k]$, only to be rotated clockwise. In other words we color graph $S[n-1,k]$ like $S_1[n,k]$ and rotate it clockwise $(i-1)$-times to get the coloring of graph $S_i[n,k]$. See Fig. 4 for the visualization of the rotated colorings.

A quick observation is that we obviously do not get a proper vertex coloring of the expanded graph $S[n,k]$ since vertices $ij\ldots j$ and $ji\ldots i$, $i \neq j$, receive the same color, that is $c_{i+j-2 \pmod k}$. By contracting the previously expanded linking edges, and getting $S[n,k]$, the merged vertex $\{i,j\}$ receives color $c_{i+j-2 \pmod k}$. Therefore, we get a proper vertex coloring of $S[n,k]$. $\square$

By setting $k=3$ we immediately get the next result:

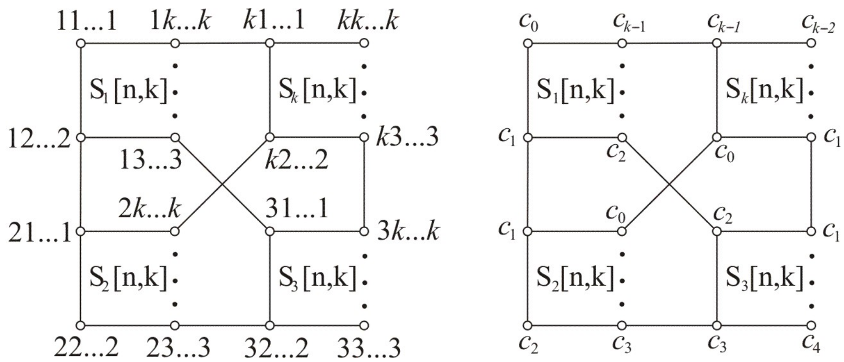

Figure 4: Vertex coloring of graph $S[n,k]$

Corollary 4.2 [25] *For any* $n \geq 1$, $\chi(S_n) = 3$.

We conclude the paper by asking what is the chromatic index and the total chromatic number of $S[n,k]$?

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