On global (strong) defensive alliances in some product graphs

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May 7, 2017

Abstract

A defensive alliance in a graph is a set S of vertices with the property that every vertex in S has at most one more neighbor outside of S than it has inside of S. A defensive alliance S is called global if it forms a dominating set. The global defensive alliance number of a graph G is the minimum cardinality of a global defensive alliance in G. In this article we study the global defensive alliances in Cartesian product graphs, strong product graphs and direct product graphs. Specifically we give several bounds for the global defensive alliance number of these graph products and express them in terms of the global defensive alliance numbers of the factor graphs.

Keywords: Defensive alliances; global defensive alliances; domination; Cartesian product graphs; strong product graph; direct product graphs.

AMS Subject Classification Numbers: 05C69; 05C70; 05C76.

1 Introduction

Alliances in graphs were first described by Kristiansen *et al.* in [10], where alliances were classified into defensive, offensive or powerful. After this seminal paper, the issue has been studied intensively. Remarkable examples are the articles [12, 13], where alliances were generalized to k-alliances, and [6], where the authors presented the first results on offensive alliances. To see more information on alliances in graphs we suggest the recent surveys [18, 19]. One of the main motivations of this study is based on the NP-completeness of computing the minimum cardinality of (defensive, offensive, powerful) alliances in graphs.

On the other hand, several graphs can be constructed from smaller and simpler components by basic operations like unions, joins, compositions, or multiplications with respect to various products, where properties of the constituents determine the properties of the composite graph. It is therefore desirable to reduce the problem of computing the graphs parameters (alliance numbers, for instance) of product graphs, to the problem of computing some parameters of the factor graphs.

Studies on alliances in product graphs have been presented in [1, 2, 14, 15, 17] where the authors presented several tight bounds for the (defensive, offensive or powerful) alliance number of Cartesian product graphs. Also, several exact values for some specific families of Cartesian product graphs were obtained in these articles. In this sense, we continue with these studies for the Cartesian product graphs and extend them to strong product graphs and direct product graphs.

Since defensive alliances defend only a single vertex at a time, Brigham *et al.* [3] introduced secure sets which are a generalization of defensive alliances. Namely, in general models, a more efficient defensive alliance should be able to defend any attack on the entire alliance or any part of it. Some general results on secure sets were presented in [4, 5], and they have also been studied on different graph products [3, 9, 16], though exact results are known only for a few family of graphs, e.g. grids, cylinders and toruses.

We begin by stating the terminology which will be used. Throughout this article, $G = (V, E)$ denotes a simple graph of order $|V| = n$. We denote two adjacent vertices u and v by $u \sim v$. Given a vertex $v \in V$, the set $N(v) = \{u \in V : u \sim v\}$ is the *neighborhood* of v, and the set $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. So, the *degree* of a vertex $v \in V$ is $\delta(v) = |N(v)|$.

For a nonempty set $S \subseteq V$, and a vertex $v \in V$, $N_S(v)$ denotes the set of neighbors v has in S, *i.e.*, $N_S(v) = S \cap N(v)$. The degree of v in S will be denoted by $\delta_S(v) = |N_S(v)|$. The *complement* of a set S in V is denoted by $\overline{S}$.

A set $S \subseteq V$ is a *dominating set* in G if for every vertex $v \in \overline{S}$, $\delta_S(v) > 0$ (every vertex in $\overline{S}$ is adjacent to at least one vertex in S). The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G [8]. An *efficient dominating set* is a dominating set $S = \{u_1, u_2, \ldots, u_{\gamma(G)}\}$ such that $N[u_i] \cap N[u_j] = \emptyset$, for every $i, j \in \{1, \ldots, \gamma(G)\}$, $i \neq j$. Examples of graphs having an efficient dominating set are the path graphs P_n, the cycle graphs C_{3k} and the cube graph Q_3.

A nonempty set $S \subseteq V$ is a *global defensive alliance* in G if S is a dominating set and

$$\delta_S(v) \geq \delta_{\overline{S}}(v) - 1, \quad \forall v \in S \tag{1}$$

The *global defensive alliance number* of G, denoted by $\gamma_d(G)$, is defined as the minimum cardinality of a global defensive alliance in G. A global defensive alliance of cardinality $\gamma_d(G)$ is called a $\gamma_d(G)$-set.

A global defensive alliance is called *strong* if

$$\delta_S(v) > \delta_{\overline{S}}(v) - 1, \quad \forall v \in S. \tag{2}$$

or, equivalently,

$$\delta_S(v) \geq \delta_{\overline{S}}(v), \quad \forall v \in S. \tag{3}$$

The *global strong defensive alliance number* of G, denoted by $\gamma_{sd}(G)$, is defined as the minimum cardinality of a global strong defensive alliance in G. A global strong defensive alliance of cardinality $\gamma_{sd}(G)$ is called a $\gamma_{sd}(G)$-set.

2 Cartesian product graphs

Given two graphs G and H with set of vertices $V_1 = \{v_1, v_2, \dots, v_{n_1}\}$ and $V_2 = \{u_1, u_2, \dots, u_{n_2}\}$, respectively, the Cartesian product of G and H is the graph $G \square H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (v_i, u_j) and (v_k, u_ℓ) are adjacent in $G \square H$ if and only if

- $v_i = v_k$ and $u_j \sim u_\ell$, or
- $v_i \sim v_k$ and $u_j = u_\ell$.

Given a set $X \subset V_1 \times V_2$ of vertices of $G \square H$, the projections of X over V_1 and V_2 are denoted by $P_G(X)$ and $P_H(X)$, respectively. Moreover, given a set $C \subset V_1$ of vertices of G and a vertex $v \in V_2$, a $G(C, v)$-*cell* in $G \square H$ is the set $C^v = \{(u, v) \in V \, : \, u \in C\}$. A v-fiber G_v is the copy of G corresponding to the vertex v of H. For every $v \in V_2$ and $D \subset V_1 \times V_2$, let D_v be the set of vertices of D belonging to the same v-fiber.

Theorem 1. *For any two graphs G and H of order n_1 and n_2, respectively, we have*

$$\gamma_d(G \square H) \leq \min\{n_1 \gamma_d(H), n_2 \gamma_d(G)\}.$$

Moreover, if G has an efficient dominating set, then

$$\gamma_d(G \square H) \geq \gamma(G)\gamma(H).$$

Proof. Let V_1 and V_2 be the vertex sets of the graphs G and H, respectively. Let A_1 and A_2 be global defensive alliances in G and H, respectively. We claim that $A = A_1 \times V_2$ is a global defensive alliance in $G \square H$. It is clear that A is a dominating set. Now, consider a vertex $(u, v) \in A$. We have the following:

$$\begin{aligned}
\delta_A(u, v) &= \delta_{A_1}(u) + \delta_H(v) \\
&\geq \delta_{\overline{A_1}}(u) - 1 + \delta_H(v) \\
&= \delta_{\overline{A_1}}(u) + \delta_H(v) - 1 \\
&\geq \delta_{\overline{A}}(u, v) - 1.
\end{aligned}$$

Thus, A is a global defensive alliance in $G \square H$. Analogously we prove that $V_1 \times A_2$ is a global defensive alliance in $G \square H$ and the proof of the upper bound is complete.

On the other hand, let $S = \{u_1, \dots, u_{\gamma(G)}\}$ be an efficient dominating set of G. Let $\Pi = \{S_1, S_2, \dots, S_{\gamma(G)}\}$ be a vertex partition of G such that $S_i = N[u_i]$. Let $\{P_1, P_2, \dots, P_{\gamma(G)}\}$ be a vertex partition of $G \Box H$, such that $P_i = S_i \times V_2$ for every $i \in \{1, \dots, \gamma(G)\}$.

Let A be a $\gamma_d(G \Box H)$-set. Now, for every $i \in \{1, \dots, \gamma(G)\}$, let $A_i = P_H(A \cap P_i)$. If A_i is not a dominating set, then there exist $w \in \overline{A_i}$ such that $N_{A_i}(w) = \emptyset$. So, since S is an efficient dominating set, vertex (u_i, w) satisfies $N_{A_i}(u_i, w) = N_A(u_i, w) = \emptyset$, which is a contradiction. Thus, A_i is a dominating set in H. Therefore we have that

$$\gamma_d(G \Box H) = |A| = \sum_{i=1}^{\gamma(G)} |A_i| \geq \sum_{i=1}^{\gamma(G)} \gamma(H) = \gamma(G)\gamma(H),$$

and the proof of the lower bound is complete. $\square$

An analogous procedure gives the following result for global strong defensive alliances.

Theorem 2. *For any two graphs G and H, without isolated vertices, of order n_1 and n_2, respectively, we have*

$$\gamma_{sd}(G \Box H) \leq \min\{n_1\gamma_d(H), n_2\gamma_d(G)\}.$$

Moreover, if G has an efficient dominating set, then

$$\gamma_{sd}(G \Box H) \geq \gamma(G)\gamma(H).$$

Proof. By using the same assumptions than in Theorem 1 we consider a vertex $(u, v) \in A$ and we have the following:

$$\begin{aligned} \delta_A(u, v) &= \delta_{A_1}(u) + \delta_H(v) \\ &\geq \delta_{\overline{A_1}}(u) - 1 + \delta_H(v) \\ &\geq \delta_{\overline{A_1}}(u) - 1 + 1 \\ &= \delta_{\overline{A}}(u, v). \end{aligned}$$

Thus, A is a global strong defensive alliance in $G \Box H$ and the upper bound is proved. The lower bound follows from the fact that $\gamma(G)\gamma(H) \leq \gamma_d(G \Box H) \leq \gamma_{sd}(G \Box H)$. $\square$

Next we improve the upper bound of Theorem 1 by introducing some restrictions in the graphs used in the product. To do so, we need to introduce some terminology. A set $S \subseteq V$ is a *total dominating set* in G if for every vertex $v \in V(G)$, $\delta_S(v) > 0$ (every vertex of G is adjacent to at least one vertex in S). The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set in G.

Theorem 3. *Let G and H be two graphs with vertex sets V_1 and V_2, respectively. If the order of H is n_2 and, any two vertices $u \in V_1$, $v \in V_2$ satisfy $\delta_H(v) \geq \delta_G(u) - 3$, then*

$$\gamma_d(G \Box H) \leq n_2\gamma_t(G).$$

Moreover, if any two vertices $u \in V_1$, $v \in V_2$ satisfy $\delta_H(v) \geq \delta_G(u) - 2$, then $\gamma_{sd}(G \Box H) \leq n_2\gamma_t(G)$.

Proof. Let A_1 be a total dominating set in G. We claim that $A = A_1 \times V_2$ is a global defensive alliance in $G \,\Box\, H$. It is clear that A is a dominating set. Now, consider a vertex $(u, v) \in A$. We have the following:

$$\begin{aligned}
\delta_A(u, v) &= \delta_{A_1}(u) + \delta_H(v) \\
&\geq \delta_{A_1}(u) + \delta_G(u) - 3 \\
&= \delta_{\overline{A_1}}(u) + 2\delta_{A_1}(u) - 3 \\
&\geq \delta_{\overline{A_1}}(u) - 1 \\
&= \delta_{\overline{A}}(u, v) - 1.
\end{aligned}$$

Thus, A is a global defensive alliance in $G \,\Box\, H$ and the bound for $\gamma_d(G \,\Box\, H)$ follows. The proof of $\gamma_{sd}(G \,\Box\, H) \leq n_2\gamma_t(G)$ is analogous to the one above. □

Let $D_1, \ldots, D_k$ be dominating sets of graph G with $|D_i| = \gamma(G)$. For all i denote with $G[D_i]$ the induced subgraph on vertices D_i. We define the number

$$\Omega(G) = \max_{1 \leq i \leq k} \{\delta(G[D_i])\}$$

as the maximum of minimum degrees of all subgraphs of G induced on any dominating set D_i, $i \in \{1, \ldots, k\}$, of size $\gamma(G)$.

Theorem 4. *Let G and H be two graphs such that $\delta(H) \geq \Delta(G) - \Omega(G) - 1$. Then*

$$\gamma_d(G \,\Box\, H) \leq \gamma(G)|V(H)|.$$

Proof. Let $D = \{u_1, \ldots, u_m\}$, $m = \gamma(G)$, be a minimum dominating set of G such that $\Omega(G) = \delta(G[D])$ and let $V(H) = \{v_1, \ldots, v_n\}$ be the set of vertices of H. Then the set $S = \{(u_1, v_1), \ldots, (u_1, v_n), (u_2, v_1), \ldots, (u_2, v_n), \ldots, (u_m, v_1), \ldots (u_m, v_n)\}$ is obviously a dominating set of $G \,\Box\, H$. The set S is also a global defensive alliance since for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ it follows that

$$\begin{aligned}
\delta_S(u_i, v_j) &= \delta_D(u_i) + \delta_H(v_j) \\
&\geq \delta(G[D]) + \delta(H) \\
&= \Omega(G) + \delta(H) \\
&\geq \Omega(G) + \Delta(G) - \Omega(G) - 1 \\
&= \Delta(G) - 1 \\
&\geq \delta_{\overline{S}}(u_i, v_j) - 1.
\end{aligned}$$

□

The above result give some interesting consequences like the following ones.

Corollary 5. *For any two integers $r, t \geq 2$ we have*

(i) $\gamma_d(P_r \,\Box\, P_t) \leq \min\left\{t\left\lceil\frac{r}{3}\right\rceil, r\left\lceil\frac{t}{3}\right\rceil\right\}$,

(ii) $\gamma_d(P_r \square C_t) \le \min\left\{t\left\lceil\frac{r}{3}\right\rceil, r\left\lceil\frac{t}{3}\right\rceil\right\}$,

(iii) $\gamma_d(C_r \square C_t) \le \min\left\{t\left\lceil\frac{r}{3}\right\rceil, r\left\lceil\frac{t}{3}\right\rceil\right\}$.

Proof. The results follow immediately from Theorem 4, from the fact that, for any integer $n \ge 2$, $\delta(P_n) = 1$, $\Delta(P_n) = 2$, $\Omega(P_n) = 0$, $\delta(C_n) = 2$, $\Delta(C_n) = 2$, and $\Omega(C_n) = 0$. Thus, if G is a path or a cycle, then the inequality $\delta(G) \ge \Delta(G) - \Omega(G) - 1$ is satisfied. □

The following lemma together with Theorem 4 leads to some equalities for the global defensive alliance numbers of some Cartesian product graphs which shows that the bound of Theorem 4 is sharp.

Lemma 6. [7, 11] *For every graph G of order n and maximum degree Δ,*

$$\gamma_d(G) \ge \left\lceil \frac{n}{\left\lfloor\frac{\Delta+1}{2}\right\rfloor + 1} \right\rceil \quad \text{and} \quad \gamma_{sd}(G) \ge \left\lceil \frac{n}{\left\lfloor\frac{\Delta}{2}\right\rfloor + 1} \right\rceil.$$

The following result is consequence of the above lemma and Theorem 4.

Corollary 7. *Let G and H be two graphs being paths or cycles of orders r and t, respectively. Then*

$$\left\lceil \frac{rt}{3} \right\rceil \le \gamma_d(G \square H) \le \gamma_{sd}(G \square H) \le \min\left\{ r\left\lceil\frac{t}{3}\right\rceil, t\left\lceil\frac{r}{3}\right\rceil \right\}.$$

Moreover, if $r \equiv 0 \pmod 3$ or $t \equiv 0 \pmod 3$, then $\gamma_d(G \square H) = \gamma_{sd}(G \square H) = \frac{rt}{3}$.

Next we continue with some other cases of Cartesian product graphs.

Proposition 8. *For any two complete graphs K_r and K_t we have*

(i) *If $r = t$, then $\gamma_d(K_r \square K_t) = \gamma_{sd}(K_r \square K_t) = r$.*

(ii) *If $|r - t| = 1$, then $\gamma_d(K_r \square K_t) = \min\{r, t\}$ and $\min\{r, t\} \le \gamma_{sd}(K_r \square K_t) \le \max\{r, t\}$.*

(iii) *If $|r - t| \ne 1$, then $\min\{r, t\} \le \gamma_d(K_r \square K_t) \le \gamma_{sd}(K_r \square K_t) \le \max\{r, t\}$.*

Proof. If $r = t$, then it is clear that every vertex of one copy of K_r or K_t, say K_r, has as much neighbors inside the copy as it has outside of the copy and also, every copy of K_r is a dominating set in $K_r \square K_t$. So, $\gamma_d(K_r \square K_t) \le r$ and $\gamma_{sd}(K_r \square K_t) \le r$. Now, since $\gamma_{sd}(K_r \square K_t) \ge \gamma_d(K_r \square K_t) \ge \gamma(K_r \square K_t) = r$, we obtain (i).

If $|r - t| = 1$, then we can suppose without loss of generality that $r = t + 1$ and let A be the set of vertices of one copy of K_t in $K_r \square K_t$. Notice that A is a dominating set in $K_r \square K_t$. Hence, for every vertex $(u, v) \in A$ we have that $\delta_A(u, v) = t - 1 = r - 2 = \delta_{\overline{A}}(u, v) - 1$. Thus, A is a global defensive alliance in $K_r \square K_t$ and $\gamma_d(K_r \square K_t) \le \min\{r, t\}$. Now, the equality for $\gamma_d(K_r \square K_t)$ in (ii) follows from the fact that $\gamma_d(K_r \square K_t) \ge \gamma(K_r \square K_t) = \min\{r, t\} = t$. Now, let B be a copy of K_r in $K_r \square K_t$. Hence, B is a dominating set in $K_r \square K_t$ and for every vertex $(u, v) \in B$, $\delta_B(u, v) = r = t + 1 > t - 1\delta_{\overline{B}}(u, v)$. So, B is a global strong defensive alliance in $K_r \square K_t$ and $\gamma_{sd}(K_r \square K_t) \le \max\{r, t\}$. The lower bound for $\gamma_{sd}(K_r \square K_t)$ in (ii) follows from the fact that $\gamma_{sd}(K_r \square K_t) \ge \gamma_d(K_r \square K_t) = \min\{r, t\}$.

On the other hand, suppose $t > r+1$. Let B be the set of vertices of one copy of K_t in $K_r \square K_t$. Notice that B is a dominating set in $K_r \square K_t$ and for every vertex $(u,v) \in B$ we have that

$$\delta_B(u,v) = t-1 > r-1 = \delta_{\overline{B}}(u,v) > \delta_{\overline{B}}(u,v) - 1.$$

Thus, B is a global (strong) defensive alliance in $K_r \square K_t$ and $\gamma_d(K_r \square K_t) \le \gamma_{sd}(K_r \square K_t) \le \max\{r,t\}$. Finally the lower bound of (iii) follows from $\gamma_{sd}(K_r \square K_t) \ge \gamma_d(K_r \square K_t) \ge \gamma(K_r \square K_t) = \min\{r,t\}$. □

3 Strong product graphs

Given two graphs G and H with set of vertices $V_1 = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V_2 = \{u_1, u_2, \ldots, u_{n_2}\}$, respectively, the strong product of G and H is the graph $G \boxtimes H = (V,E)$, where $V = V_1 \times V_2$ and two vertices (v_i, u_j) and (v_k, u_ℓ) are adjacent in $G \boxtimes H$ if and only if

- $v_i = v_k$ and $u_j \sim u_\ell$, or
- $v_i \sim v_k$ and $u_j = u_\ell$, or
- $v_i \sim v_k$ and $u_j \sim u_\ell$.

Theorem 9. *For any two graphs G and H of order r and t, respectively, we have*

$$\gamma_d(G \boxtimes H) \le \min\{r\gamma_d(H), t\gamma_d(G)\}.$$

Proof. Let V_1 and V_2 be the vertex set of G and H, respectively. Let $S_1 \subset V_1$ be a global defensive alliance in G and let $A = A_1 \times V_2$. Since S_1 is a dominating set in G, we have that A is a dominating set in $G \boxtimes H$. Also, for every vertex $(u,v) \in A$ we have

$$\begin{aligned}\delta_A(u,v) &= \delta_{A_1}(u) + \delta(v) + \delta(v)\delta_{A_1}(u)\\ &\ge \delta_{\overline{A_1}}(u) - 1 + \delta(v) + \delta(v)(\delta_{\overline{A_1}}(u) - 1)\\ &= \delta_{\overline{A_1}}(u) + \delta(v)\delta_{\overline{A_1}}(u) - 1\\ &= \delta_{\overline{A}}(u,v) - 1.\end{aligned}$$

So, A is a global defensive alliance in $G \boxtimes H$. Analogously we prove that $V_1 \times A_2$ is also a global defensive alliance in $G \boxtimes H$, where A_2 is a global defensive alliance in H. Therefore the result follows. □

Let G be the graph of order six obtained by joining with an edge the centers of two star graphs of order three. Notice that $\gamma_d(G) = 2$. Hence, we have that $\gamma_d(G \boxtimes K_2) = 4 = \min\{6 \cdot 1, 2 \cdot 2\}$. Thus, the bound of Theorem 9 is tight. Another case in which this bound is tight is the strong product of two complete graphs K_r and K_t of even orders, where we have that $\frac{rt}{2} = \gamma_d(K_{rt}) = \gamma_d(K_r \boxtimes K_t) = r\frac{t}{2} = t\frac{r}{2}$.

Next, we study some particular cases of strong product graphs. To do so we need the following lemma.

Lemma 10. [7] *For any integer* $n \geq 2$, $\gamma_d(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor$, $\gamma_d(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$, $\gamma_d(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$ *if* $n \not\equiv 2 \pmod 4$ *and* $\gamma_d(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor - 1$ *if* $n \equiv 2 \pmod 4$.

Proposition 11. *For any two integers* $r, n \geq 4$ *we have*

$$\frac{rn}{2} \leq \gamma_d(C_r \boxtimes K_n) \leq \min\left\{ r\left\lfloor \frac{n+1}{2} \right\rfloor, n\left(\left\lfloor \frac{r}{2} \right\rfloor + \left\lceil \frac{r}{4} \right\rceil - \left\lfloor \frac{r}{4} \right\rfloor \right) \right\}.$$

Moreover, if n *is an even number, then* $\gamma_d(C_r \boxtimes K_n) = \frac{rn}{2}$.

Proof. The upper bound follows directly from Theorem 9 and Lemma 10. Let $V_1 = \{u_0, u_1, \ldots, u_{r-1}\}$ be the vertex set of C_r (vertices with consecutive indices are adjacent in C_r and operations with indices is considered modulo r) and let $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n. Let S be a global defensive alliance of minimum cardinality in $C_r \boxtimes K_n$ and let $(u_i, v_j) \in S$. Hence we have that $\delta_S(u_i, v_j) \geq \delta_{\overline{S}}(u_i, v_j) - 1$, which is equivalent to

$$\delta_S(u_i, v_j) \geq \frac{\delta_{C_r \boxtimes K_n}(u_i, v_j) - 1}{2} = \frac{3n-2}{2}. \tag{4}$$

Now, for every $i \in \{0, \ldots, r-1\}$, let $S_i = S \cap (\{u_{i-1}, u_i, u_{i+1}\} \times V_2)$. It is clear that for every $i \in \{1, \ldots, n\}$, $S_i \neq \emptyset$. So, from inequality 4 we obtain that $|S_i| \geq \frac{3n-2}{2} + 1 = \frac{3n}{2}$. Thus, we have that

$$|S| \geq \frac{1}{3} \sum_{i=0}^{r-1} |S_i| \geq \frac{1}{3} \sum_{i=0}^{r-1} \frac{3n}{2} = \frac{rn}{2}$$

and the lower bound is proved.

Notice that, if n is even, then $\left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n}{2}$. Thus, the upper bound is $\gamma_d(C_r \boxtimes K_n) \leq \frac{rn}{2}$ and the equality follows for n being even. □

The proof of the next result is relatively similar to the above proof.

Proposition 12. *For any two integers* $r, n \geq 4$ *we have*

$$\frac{n(r-2)}{2} \leq \gamma_d(P_r \boxtimes K_n) \leq \begin{cases} \min\left\{ r\left\lfloor \frac{n+1}{2} \right\rfloor, n\left(\left\lfloor \frac{r}{2} \right\rfloor + \left\lceil \frac{r}{4} \right\rceil - \left\lfloor \frac{r}{4} \right\rfloor \right) \right\}, & \text{if } n \not\equiv 2 \pmod 4, \\ \min\left\{ r\left\lfloor \frac{n+1}{2} \right\rfloor, n\left(\left\lfloor \frac{r}{2} \right\rfloor + \left\lceil \frac{r}{4} \right\rceil - \left\lfloor \frac{r}{4} \right\rfloor - 1 \right) \right\}, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. The upper bounds follow directly from Theorem 9 and Lemma 10. Let $V_1 = \{u_1, u_1, \ldots, u_r\}$ be the vertex set of P_r (vertices with consecutive indices are adjacent in P_r) and let $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n. Let S be a global defensive alliance of minimum cardinality in $P_r \boxtimes K_n$ and let $(u_i, v_j) \in S$. Hence, we have that $\delta_S(u_i, v_j) \geq \delta_{\overline{S}}(u_i, v_j) - 1$, and if $i \neq 1$ or r, then this is equivalent to

$$\delta_S(u_i, v_j) \geq \frac{\delta_{P_r \boxtimes K_n}(u_i, v_j) - 1}{2} = \frac{3n-2}{2}. \tag{5}$$

Now, for every $i \in \{1, \ldots, r-1\}$, let $S_i = S \cap (\{u_{i-1}, u_i, u_{i+1}\} \times V_2)$. It is clear that for every $i \in \{1, \ldots, n\}$, $S_i \neq \emptyset$. So, from inequality 5 we obtain that $|S_i| \geq \frac{3n-2}{2} + 1 = \frac{3n}{2}$. Thus, we have that $|S| \geq \frac{1}{3} \sum_{i=2}^{r-1} |S_i| \geq \frac{1}{3} \sum_{i=2}^{r-1} \frac{3n}{2} = \frac{n(r-2)}{2}$ and the lower bound is proved. □

4 Direct product graphs

Given two graphs G and H with set of vertices $V_1 = \{v_1, v_2, \dots, v_{n_1}\}$ and $V_2 = \{u_1, u_2, \dots, u_{n_2}\}$, respectively, the direct product of G and H is the graph $G \times H = (V, E)$, where $V = V_1 \times V_2$ and two vertices (v_i, u_j) and (v_k, u_ℓ) are adjacent in $G \times H$ if and only if $v_i \sim v_k$ and $u_j \sim u_\ell$.

Theorem 13. *For any two graphs G and H of order n_1 and n_2, respectively, we have*

$$\gamma_{sd}(G \times H) \le \min\{n_1 \gamma_{sd}(H), n_2 \gamma_{sd}(G)\}.$$

Proof. Let V_1 and V_2 be the vertex sets of the graphs G and H, respectively. If A_1 and A_2 are global strong defensive alliances in G and H, respectively, then we claim that $A = A_1 \times V_2$ is a global strong defensive alliance in $G \times H$. Notice that A is a dominating set. We consider a vertex $(u, v) \in A$. So, by inequality 3 we have that

$$\delta_A(u, v) = \delta_{A_1}(u)\delta(v) \ge \delta_{\overline{A_1}}(u)\delta(v) \ge \delta_{\overline{A}}(u, v).$$

Thus, A is a global strong defensive alliance in $G \times H$. Analogously, one can proved that $V_1 \times A_2$ is a global strong defensive alliance in $G \times H$ and the proof is complete. □

By using similar techniques like the ones used in Propositions 11 and 12 for strong product graphs we can obtain the following lower bounds.

Proposition 14. *For any two integers $r, n \ge 4$ we have*

$$\gamma_{sd}(C_r \times K_n) \ge \frac{rn}{3} + \frac{r}{3} \quad \textit{and} \quad \gamma_{sd}(P_r \times K_n) \ge \frac{n(r-2)}{3} + \frac{r-2}{3}.$$

Proof. Let $V_1 = \{u_0, u_1, \dots, u_{r-1}\}$ be the vertex set of C_r (vertices with consecutive indices are adjacent in C_r and operations with the subindices are done modulo r) and let $V_2 = \{v_1, v_2, \dots, v_n\}$ be the vertex set of K_n. Let S be a global strong defensive alliance of minimum cardinality in $C_r \times K_n$ and let $(u_i, v_j) \in S$. Hence, we have that

$$\delta_S(u_i, v_j) \ge \frac{\delta_{C_r \times K_n}(u_i, v_j)}{2} = \frac{2n}{2} = n. \tag{6}$$

Now, for every $i \in \{0, \dots, r-1\}$, let $S_i = S \cap (\{u_{i-1}, u_i, u_{i+1}\} \times V_2)$. It is clear that for every $i \in \{1, \dots, n\}$, $S_i \ne \emptyset$. So, from inequality 6 we have that $|S_i| \ge n + 1$. Thus, we obtain

$$|S| \ge \frac{1}{3} \sum_{i=0}^{r-1} |S_i| \ge \frac{1}{3} \sum_{i=0}^{r-1} (n+1) = \frac{rn}{3} + \frac{r}{3}.$$

Now, if $V_1 = \{u_1, u_1, \dots, u_r\}$ is the vertex set of P_r (vertices with consecutive indices are adjacent in P_r), then by using a similar procedure we have that

$$|S| \ge \frac{1}{3} \sum_{i=2}^{r-1} |S_i| \ge \frac{1}{3} \sum_{i=2}^{r-1} (n+1) = \frac{n(r-2)}{3} + \frac{r-2}{3}.$$

□

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