The b-chromatic number and related topics-a survey

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Abstract

The b-chromatic number of a graph G is the largest integer k such that G admits a proper k-coloring in which every color class contains at least one vertex that has a neighbor in each of the other color classes. In this survey we present the most important results on b-colorings, b-chromatic number and related topics.

Key words: b-colorings, b-chromatic number, b-chromatic index, b-continuous graphs, b-critical graphs, b-perfect graphs

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1 Introduction

Let $G = (V(G), E(G))$ be a graph. We use n for the order and m for the size of G. As usual we denote by $\Delta(G)$ and $\delta(G)$ the maximum and the minimum degree of a graph, respectively, while $d_G(v)$ or simply $d(v)$, if there can be no confusion, means the degree of a vertex. The independence number of G is denoted by $\alpha(G)$ and its clique number is $\omega(G)$. A *clique partition* for a graph G is any partition of $V(G)$ into subsets $C_1, \ldots, C_k$ in such a way that the subgraph of G induced by C_i is a clique, for each i. We denote by $\theta(G)$ the minimum number of subsets in a clique partition of the graph G and call it the *clique cover number* of G. Graph $\overline{G}$ is the complement of G and $\nu(G)$ is the cardinality of a maximum matching of G. The *vertex-connectivity* of a connected graph G, denoted by $\kappa(G)$, is the minimum size of a subset $X \subseteq V(G)$ such that $G - X$ is either disconnected or a graph with only one vertex. The *edge-connectivity* of a connected graph G with at least two vertices, denoted by $\lambda(G)$, is the minimum size of a subset $Y \subseteq E(G)$ such that $G - Y$ is disconnected. The edge-connectivity of the vertex graph is defined as 0. It always holds that $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

A function $c : V(G) \to \{1, \ldots, k\}$ is called a *proper vertex coloring* (with k colors) if $c(u) \neq c(v)$ for every edge $uv \in E(G)$. The minimum number $\chi(G)$ for which there exists a proper vertex coloring (with $\chi(G)$ colors) is called the *chromatic number* of a graph G. Every proper vertex coloring c yields a partition of $V(G)$ into sets $V_i = \{u \in V(G) \,|\, c(u) = i\}$, for every $i \in \{1, \ldots, k\}$, called *color classes* of c.

Let $F(G)$ be the set of all proper colorings of G and let $c \in F(G)$. A vertex v of G with $c(v) = i$ is a b-*vertex* (of color i), if there exists a neighbor u of v with $c(u) = j$ for every $j \neq i$. If a vertex v with $c(v) = i$ is not a b-vertex, then we can recolor v with some color that is not represented in the neighborhood of v to obtain a slightly different coloring. Hence, if there exists no b-vertex of color i, then we can recolor every vertex colored with i and we obtain a new coloring $c_i : V(G) \to \{1, \ldots, k\} - \{i\}$. Clearly c_i is a proper $(k-1)$-coloring of G. Next we define the relation $\lhd$ on $F(G) \times F(G)$. We say that c_1 is in relation $\lhd$ with c, $c_1 \lhd c$, if c_1 can be obtained from c by recoloring every vertex of one fixed color class of c. Clearly, $\lhd$ is asymmetric. Let the relation $\prec$ be the transitive closure of $\lhd$. We get a strict partial ordering $\prec$. Since there are finitely many different proper colorings of a graph G, this ordering has some minimal elements. The maximum number of colors used in a minimal element of ordering $\prec$ is called the *b-chromatic number* $\varphi(G)$, in contrast with the chromatic number $\chi(G)$, which is the minimum number of colors used in a minimal element of $\prec$.

Since the introductory paper by Irving and Manlove in 1999 [44], there was a big interest in the community for this invariant, as can be seen from references of this survey. Here we try to present a dense overview about the most important results. Also, b-colorings were probably the main motivation for a broader set of colorings, called recoloring-resistant colorings which were introduced by Pedersen and Rautenbach in [67]. We start with some bounds for the b-chromatic number, followed by a section on regular graphs. The b-chromatic number of some graph classes will be presented in the fourth section. Next section presents some closely related concepts which have origins in b-colorings and in the b-chromatic number, such as $m(G)$-tight graphs, b-critical graphs, b-perfect graphs and b-continuous graphs. A section on graph operations follows. In some sense a special kind of operation—the b-chromatic number of a line graph called as usual the b-chromatic index—is covered in Section 7 and in the last section we give a discussion on complexity and algorithmic aspects.

2 Bounds

Since every b-coloring is a proper coloring, we obtain that the chromatic number $\chi(G)$ is a lower bound for $\varphi(G)$. For the upper bound notice that every color class must have a b-vertex and moreover a b-vertex can have at most $\Delta(G)$ different colors in its neighborhood. The only additional color which is possible, is the color of a b-vertex itself. Therefore the trivial upper bound for $\varphi(G)$ is $\Delta(G)+1$. Hence, we have the following bounds

$$\chi(G) \leq \varphi(G) \leq \Delta(G)+1.$$

The mentioned upper bound can sometimes be reduced. For a b-coloring of a graph one needs enough vertices of high degree. Indeed, if we have a b-coloring of G with k colors, then we need at least k vertices of degree at least $k-1$ to ensure enough b-vertices. Suppose that vertices $v_1,\ldots,v_n$ of G are ordered such that $d(v_1) \geq \ldots \geq d(v_n)$ holds, where $d(v_i)$ represents the degree of v_i. The invariant $m(G) = \max\{i \,|\, i-1 \leq d(x_i)\}$ has surprisingly no special name, but it is very handy dealing with the b-chromatic number. Already in [44] Irving and Manlove observed the following.

Proposition 2.1 [44] *If G is a graph, then $\varphi(G) \leq m(G)$.*

Clearly, $m(G) = \Delta(G)+1$ for a regular graph G and each vertex can be a b-vertex in such a graph. This is one of the main reasons that the b-chromatic number of regular graphs draws a special attention, see Section 3.

The chain of relations between $\varphi(G)$ and some other coloring invariants as fall chromatic number $\chi_f(G)$, fall achromatic number $\Psi_f(G)$, partial Grundy number $\partial\Gamma(G)$ and achromatic number $\Psi(G)$ is presented in [21]. They are as follows

$$\chi(G) \leq \chi_f(G) \leq \Psi_f(G) \leq \varphi(G) \leq \partial\Gamma(G) \leq \Psi(G).$$

Later it was shown in [4] that above inequalities are strict.

In [1, 2, 9, 55, 57] one can find a large number of (lower and upper) bounds on the b-chromatic number. Especially in [1] the variety of them is really impressing. Since many of them are restricted to some special graph classes or even only to some graph families, we present here a selection of them. We start to compare the b-chromatic number with the size and the order of a graph.

Proposition 2.2 [55] *For any graph G of size m, we have $\varphi(G) \leq \frac{1}{2}+\sqrt{2m+\frac{1}{4}}$.*

The bound in Proposition 2.2 follows immediately from

$$2m = \sum_{i=1}^{n} d(v_i) \geq \varphi(G)(\varphi(G)-1),$$

where $\{v_1,\ldots,v_n\}$ are the vertices of graph G.

Theorem 2.3 [55] *If G is a graph, then $\varphi(G) + \varphi(\overline{G}) \leq n + 1$.*

The bound in Theorem 2.3 is achieved for example by the join of the stable set S_p and the complete graph K_{n-p}. The same example also gives equality for next general upper bound.

Proposition 2.4 [55] *For a connected graph G, $\varphi(G) \leq n + 1 - \alpha(G)$.*

Theorem 2.5 [58] *If G is the complement of a bipartite graph, then $\varphi(G) \leq \frac{4\omega(G)}{3}$.*

Kouider and Zaker showed in [58] that for any ω divisible by 3 there exists a bipartite graph such that for its complement graph G with clique number $\omega(G) = \omega$ the equality $\varphi(G) = \frac{4\omega(G)}{3}$ holds.

Theorem 2.6 [1] *For every non-complete graph G, $\varphi(G) \leq \left\lceil \frac{n+\omega(G)}{2} \right\rceil - 1$.*

The bound in Theorem 2.6 is sharp for the cycle C_5 and every complete bipartite graph $K_{n,n}$ where a perfect matching is removed.

Theorem 2.7 [1] *For every graph G with the clique cover number $\theta(G) \leq t$, $\varphi(G) \leq \left\lfloor \frac{t\omega(G)+(t-1)n}{2t-1} \right\rfloor$.*

The result in Theorem 2.7 improves a similar result from Kouider and Zaker in [58]. Their upper bound $\left\lfloor \frac{t^2\omega(G)}{2t-1} \right\rfloor$ is not smaller than the bound in Theorem 2.7 when $\theta(G) \leq t$. Moreover, Kouider and Zaker proved in [58] that their upper bound is sharp.

Next, we present two more results from Alkhateeb and Kohl in [1].

Theorem 2.8 [1] *For every graph G, $\varphi(G) \leq \left\lceil n - \frac{2\nu(\overline{G})}{3} \right\rceil$.*

Theorem 2.9 [1] *For every graph G, $\varphi(G) \leq \left\lfloor \frac{2n-\Delta(G)-\delta(G)-3}{3n-2\Delta(G)-\delta(G)-4} n \right\rfloor$.*

In [1] an example is given amongst the complement graphs of 3-regular connected graphs, which attains the bounds in Theorems 2.8 and 2.9.

The lower bound from the next result is from [2] and the upper bound from [9]. Also in [2], graphs are discussed for which the equality holds.

Theorem 2.10 [2, 9] *For any connected graph G with $n \geq 5$ vertices and for any $v \in V(G)$,*

$$\varphi(G) - \left(\left\lceil \frac{n}{2} \right\rceil - 2 \right) \leq \varphi(G - v) \leq \varphi(G) + \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

We end this section with one upper and two lower bounds for bipartite graphs from Kouider, Valencia-Pabon and Zacker [57, 58]. For the upper bound the *biclique number* of G is the minimum number of disjoint complete bipartite subgraphs which cover vertices of G.

Theorem 2.11 [58] *Let G be a bipartite graph with partition $V(G) = A \cup B$ and biclique number c. Then $\varphi(G) \le \left\lfloor \frac{n-c+4}{2} \right\rfloor$.*

It was shown in [58] that for any integer $t \ge 3$, there exists a bipartite graph G with $n = 3t-4$ vertices and biclique number $c = t-1$ such that $\varphi(G) = t = \left\lfloor \frac{n-c+4}{2} \right\rfloor$.

Theorem 2.12 [57] *Let G be a connected bipartite graph with partition $V(G) = A \cup B$. If there are subsets $A_0 \subseteq A$ and $B_0 \subseteq B$ such that*

1. *the induced subgraph $G[A_0 \cup B_0]$ is isomorphic to $K_{p,p} - M$ for some positive integer p and a perfect matching M of $K_{p,p}$, and*
2. *$A - A_0$ does not dominate B_0 or $B - B_0$ does not dominate A_0,*

then $\varphi(G) \ge p$.

Before we present a second lower bound from [57] we need some additional definitions. Let G be a connected bipartite graph with partition $V(G) = A \cup B$. An edge $xy \in E(G)$ is a *dominating edge* in G if $N(x) \cup N(y) = A \cup B = V(G)$. By $\widetilde{N}(u)$, $u \in A$ (resp. $u \in B$), we denote the set of all non-neighbors of u in B (resp. in A). Let $S = (a_1, B_1), \ldots, (a_p, B_p)$ be a sequence with $a_i \in A$, $B_i \subset B$, where $a_i \ne a_j$ and $B_i \cap B_j = \emptyset$ whenever $i \ne j$, constructed as follows. Vertex $a_1 \in A$ is such that $\left|\widetilde{N}(a_1)\right| = \min\left\{\left|\widetilde{N}(a_i)\right| \mid a_i \in A\right\}$ and set $B_1 = \widetilde{N}(a_1)$. When we have constructed (a_i, B_i), we choose (a_{i+1}, B_{i+1}) as follows:

1. $B_j \not\subseteq \widetilde{N}(a_{i+1})$, for all $j \le i$.
2. $\left|\widetilde{N}(a_{i+1}) - \cup_{j=1}^{i} B_j\right|$ is minimum and not equal to zero. Set $B_{i+1} = \widetilde{N}(a_{i+1}) - \cup_{j=1}^{i} B_j$.

We call S a *good-sequence* of size p for G.

Theorem 2.13 [57] *Let G be a connected bipartite graph with partition $V(G) = A \cup B$ without dominating edges. If $(a_1, B_1), \ldots, (a_p, B_p)$ is a maximal good-sequence of size $p \ge 2$ for G, then $\varphi(G) \ge p$.*

3 The b-chromatic number of regular graphs

Regular graphs play an interesting and important role for the b-chromatic number since for every regular graph G equality $m(G) = \Delta(G) + 1$ holds. It was proved in [60] that $\varphi(G) = \Delta(G) + 1$ for a d-regular graph G on at least d^4 vertices. Even though this is more or less a trivial bound on the number of vertices, it shows that there are only a finite number of d-regular graphs with b-chromatic number strictly less than $\Delta(G)+1$. Nevertheless, case analysis shows that the bound d^4 is probably much smaller. Indeed, this bound was lowered by Cabello and Jakovac [13].

Theorem 3.1 [13] *Let G be a d-regular graph with at least $2d^3$ vertices. Then $\varphi(G) = d + 1$.*

The idea of the proof is using Hall's Marriage Theorem. By modifying this idea, the theorem was later slightly improved by El Sahili et al. [29]. In the same paper, another proof is given without the use of matchings.

Theorem 3.2 [29] *Let G be a d-regular graph, $d \geq 7$, such that $|V(G)| \geq 2d^3 + 2d - 2d^2$, then $\varphi(G) = d + 1$.*

Even though this is a much better bound than d^4, it was conjectured in [13] that it can be lowered to $c \cdot d^2$, where $c > 1$.

Knowing that for a given positive integer $d \geq 2$ there are only a finite number of connected d-regular graphs G for which $\varphi(G) \leq \Delta(G)$, it would be interesting to know the list of such graphs. For $d = 2$ this list is trivial since only C_4 is on this list. It is well known that the Petersen graph is on the list of 3-regular (cubic) graphs as shown in [11]. The full list of cubic graphs was later given by Jakovac and Klavžar, who proved the following theorem.

Theorem 3.3 [45] *Let G be a connected cubic graph. Then $\varphi(G) = 4$ unless G is P, $K_3 \,\square\, K_2$ (left graph of Fig. 1), $K_{3,3}$, or G_1 (right graph of Fig. 1). In these cases, $\varphi(P) = \varphi(K_3 \,\square\, K_2) = \varphi(G_1) = 3$ and $\varphi(K_{3,3}) = 2$.*

Since the cycle C_4 is one of the examples for which $\varphi(C_4) < \Delta(C_4) + 1$, it is no surprise that it is challenging to find the b-chromatic number for a graph which contains graph C_4 as a subgraph. Therefore, quite some results on the b-chromatic number of regular graphs which forbid the subgraph C_4 and achieve the upper bound $d+1$ were given. First, we mention some with respect to the diameter of the graph.

Theorem 3.4 [13] *Let G be a d-regular graph with no 4-cycles and* $\mathrm{diam}(G) \geq d$. *Then $\varphi(G) = d + 1$.*

In [68] Shaebani lowered the bound on the diameter and showed that it is not dependent on the degree of a graph.

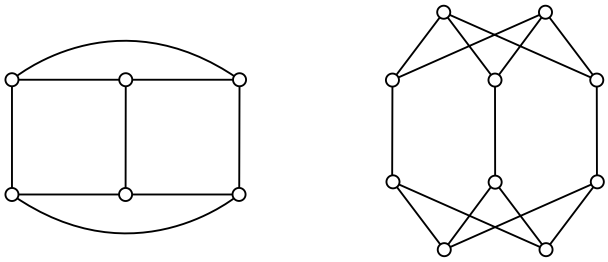

Figure 1: The prism $K_3 \,\square\, K_2$ and the graph G_1

Theorem 3.5 [68] *Let G be a d-regular graph that contains no 4-cycle. If* $\mathrm{diam}(G) \geq 6$*, then* $\varphi(G) = d + 1$.

There are also some other results on the b-chromatic number which forbid the 4-cycle as a subgraph and were derived in [26, 29]

Theorem 3.6 [26] *Let G be a d-regular graph, $d \geq 7$, containing neither 4-cycles nor 6-cycles. Then $\varphi(G) = d + 1$.*

Theorem 3.7 [26] *Let G be a d-regular graph, $d \geq 7$, with no cycle of order 4 such that there exists a vertex x in G with $N(x) = \{x_1, \ldots, x_d\}$ that satisfies the following conditions:*

1. $|\{v \in N(x_i) \,|\, N(v) \cap (\cup_{j \neq i} N(x_j)) \neq \emptyset\}| \leq \lceil \frac{d-1}{2} \rceil - 1$*, for every i, $1 \leq i \leq d$,*
2. $|N(v) \cap (\cup_{j \neq i} N(x_j))| \leq \lceil \frac{d-1}{2} \rceil - 1$ *for every $v \in N(x_i)$ and for all i, $1 \leq i \leq d$.*

Then $\varphi(G) = d + 1$.

Theorem 3.8 [29] *Let G be a d-regular graph, $d \geq 7$, with no 4-cycles. If $|V(G)| \geq d^3 + d$, then $\varphi(G) = d + 1$.*

Another approach to the b-chromatic number of regular graphs is through the size of their *girth*. The girth in graph G is the size of the smallest cycle in the graph and is denoted by $g(G)$. As it turns out, every regular graph G with large girth has b-chromatic number equal to $\Delta(G) + 1$. This was first proved by Kouider.

Theorem 3.9 [54] *Every d-regular graph G with girth $g(G) \geq 6$ has a b-coloring with $d + 1$ colors.*

It is therefore interesting to see what happens when the girth of the graph is smaller. One of the first results considering smaller girth of a regular graph was proved by El Sahili and Kouider.

Theorem 3.10 [27] *If G is a d-regular graph with girth $g(G) \geq 5$ and G contains no C_6, then $\varphi(G) = d + 1$.*

A new and similar result on this matter was derived by Blidia et al. in [11] who show that girth 5 is enough if the degree of vertices is not too big excluding the Petersen graph. This theorem is also a complement for small values of the degree to Theorems 3.6, 3.7 and 3.8.

Theorem 3.11 [11] *Let G be a d-regular graph with girth $g(G) \geq 5$, different from the Petersen graph, and with $d \leq 6$. Then $\varphi(G) = d + 1$.*

Surprisingly, vertex- and edge-connectivity are tightly related to the fact whether the b-chromatic number of a d-regular graph equals $d + 1$. Two results that deal with the vertex-connectivity were proved by Shaebani in [68].

Theorem 3.12 [68] *Let G be a d-regular graph that contains no 4-cycle. If $\kappa(G) \leq \frac{d+1}{2}$, then $\varphi(G) = d + 1$.*

Theorem 3.13 [68] *Let G be a d-regular graph that contains no 4-cycle. If $\kappa(G) < \frac{3d-3}{4}$, then $\min\{2(d - \kappa(G) + 1), d + 1\} \leq \varphi(G)$. Besides, if there exists a subset U of $V(G)$ such that $|U| = \kappa(G)$ and $G - U$ has at least four connected components, then $\varphi(G) = d + 1$. Moreover, if $\kappa(G) < \frac{2d-1}{3}$ and there exists a subset U of $V(G)$ such that $|U| = \kappa(G)$ and $G - U$ has at least three connected components, then $\varphi(G) = d + 1$. If the girth of G is 5, $\frac{3d-3}{4}$ and $\frac{2d-1}{3}$ can be replaced by $\frac{3d}{4}$ and $\frac{2d+1}{3}$, respectively.*

The result on edge-connectivity is a little more involved since one needs the definition of super-edge-connectivity [70]. For a nonempty proper subset S of $V(G)$, the edge-cut is the set of edges having one endpoint in S and the other endpoint in $V(G) - S$. An *edge-cut* is trivial if it is equal to the set of all edges incident to a single vertex. A d-regular graph is *super-edge-connected* if every minimum edge-cut is a trivial edge-cut.

Theorem 3.14 [70] *Let G be a d-regular graph that contains no 4-cycle. If G is not super-edge-connected, then $\varphi(G) = d + 1$.*

There exist regular graphs with small b-chromatic number, e.g. the complete bipartite graph $K_{n,n}$, whose b-chromatic number equals $\varphi(K_{n,n}) = 2$. This graph has girth 4. The result of Cabello and Jakovac [13], which was later improved by Shaebani [68], shows that this is not possible for regular graphs with girth 5. Namely, it is shown that the parameter $\varphi(G)$ is bounded from below by a linear function of the degree.

Theorem 3.15 [13] *Let G be a d-regular graph with girth 5. Then $\varphi(G) \geq \left\lfloor \frac{d+1}{2} \right\rfloor$.*

Theorem 3.16 [68] *If G is a d-regular graph that contains no 4-cycles, then* $\varphi(G) \geq \left\lfloor \frac{d+3}{2} \right\rfloor$. *If G contains a triangle, then* $\varphi(G) \geq \left\lfloor \frac{d+4}{2} \right\rfloor$.

For the purpose of studying the b-chromatic number of d-regular graphs with no 4-cycle and girth at least 5 another invariant, which closely resembles the b-chromatic number, was introduced in [26]. It is called the *f-chromatic vertex number* of a d-regular graph G and is the maximum number of b-vertices of distinct colors in a $(d+1)$-coloring of G. This invariant is denoted by $f(G)$. It is important to note that this new invariant is a lower bound for the b-chromatic number of d-regular graphs.

Theorem 3.17 [26] *Let G be a d-regular graph, then* $\varphi(G) \geq f(G)$.

One can use the result in Theorem 3.17 to establish lower bounds of $\varphi(G)$. Namely, every lower bound of $f(G)$ is also a lower bound of $\varphi(G)$. Theorem 3.16 by Shaebani was slightly improved for even d with the following result.

Theorem 3.18 [26] *Let G be a d-regular graph, $d \geq 7$, with no cycle of order 4. Then* $f(G) \geq \left\lceil \frac{d-1}{2} \right\rceil + 2$.

With the help of the f-chromatic vertex number another results was given by El Sahili et al. with respect to the girth and diameter which closely resembles those of Cabello and Jakovac in [13] and Sheabani in [68].

Theorem 3.19 [26] *Let G be a d-regular graph, $d \geq 7$, with $g(G) = 5$ and $\mathrm{diam}(G) = 5$. Then* $f(G) \geq \left\lceil \frac{d-1}{2} \right\rceil + 4$.

We end this section with one of the most important conjectures regarding the b-chromatic number of regular graphs which was first posted by El Sahili and Kouider and later reformulated by Blidia et al. (excluding the Petersen graph).

Conjecture 3.20 [11, 27] *Every d-regular graph with girth at least 5, different from the Petersen graph, has a b-coloring with $d+1$ colors.*

4 The b-chromatic number of some graph classes

As usual for NP-complete problems (see Section 8) one tries to find exact values for some special graph classes, which often means a polynomial algorithm for determining their exact value. The cornerstone for all such results was done in introductory paper by Irving and Manlove [44] for trees. All other results adopt their method to other classes.

4.1 Trees

The trees have been settled already in the pilot paper for the b-chromatic number by Irving and Manlove in [44]. Since it is a model for many other results we present it in more detail. A vertex v of a tree T such that $d(v) \geq m(T) - 1$ is called a *dense* vertex of T.

Definition 4.1 [44] *A tree T is called* pivoted *if it has exactly $m(T)$ dense vertices and a vertex v such that:*

1. *v is not dense.*
2. *Each dense vertex is adjacent either to v or to a dense vertex adjacent to v.*
3. *Any dense vertex adjacent to v and to another dense vertex has degree $m(T)-1$.*

We call v the pivot *of T.*

Theorem 4.2 [44] *If T is a pivoted tree, then $\varphi(T) = m(T) - 1$.*

Before one can show that all the other trees have $\varphi(T) = m(T)$ the following two definitions are needed.

Definition 4.3 [44] *Let $T = (V, E)$ be a tree and let V' be the set of dense vertices of T. Suppose that V'' is a subset of V' of cardinality $m(T)$. Then V''* encircles *some vertex $v \in V - V''$ if:*

1. *Each vertex in V'' is adjacent either to v or to some vertex in V'' adjacent to v.*
2. *Any vertex in V'' adjacent to v and to another vertex in V'' has degree $m(T)-1$.*

We refer to v as an encircled vertex with respect to V''

Definition 4.4 [44] *Let $T = (V, E)$ be a tree and let V' be the set of dense vertices of T. Suppose that V'' is a subset of V' of cardinality $m(T)$. Then V'' is a* good set *with respect to T if:*

1. *Set V'' does not encircle any vertex in $V - V''$.*
2. *Any vertex $u \in V - V''$ with $d(u) \geq m(T)$ is adjacent to some $v \in V''$ with $d(v) = m(T) - 1$.*

Theorem 4.5 [44] *If T is a non-pivoted tree, then T has a good set. Moreover, a good set can be constructed in polynomial time.*

Moreover, a tree can be colored in such a way that every vertex of a good set is a b-vertex for a different color. This yields the next result which describes the b-chromatic number of all non-pivoted trees.

Theorem 4.6 [44] *If T is a non-pivoted tree, then $\varphi(T) = m(T)$.*

One can clearly check in polynomial time if a tree T is pivoted or not and this yields a polynomial algorithm for computing $\varphi(T)$.

4.2 Cactus graphs

A class of graphs that is close to trees are cacti. A graph G is a *cactus* if G does not contain two cycles that share an edge. They have been treated in [15] and it is not surprising that the methods used here are similar to the ones for trees. However, there are also important differences. Since we do not wish to extend the number of pages of this survey too much, we do not present all the details. Thus see [15] for the exact definition of *cacti-pivoted* graphs (called just pivoted cactus there).

Theorem 4.7 [15] *If G is a is a cacti-pivoted connected cactus with $m(G) \geq 7$, then $\varphi(G) = m(G) - 1$.*

Theorem 4.8 [15] *If G is a non-cacti-pivoted cactus with $m(G) \geq 7$, then $\varphi(G) = m(G)$.*

4.3 Outerplanar graphs

A planar graph G is called *outerplanar*, if there exists a planar drawing of G such that all vertices lie on one face. Cactus graphs are clearly outerplanar. Maffray and Silva generalized in [65] the approach for trees and cacti graphs to outerplanar graphs.

Theorem 4.9 [65] *Let G be an outerplanar graph with girth at least 8. If G has no good set, then $\varphi(G) = m(G) - 1$.*

Theorem 4.10 [65] *Let G be an outerplanar graph with girth at least 8. If G has a good set, then $\varphi(G) = m(G)$.*

Notice that the original definition of a good set is used again in above theorems in contrast to Theorems 4.7 and 4.8. The reason for this is the girth of a graph which is large enough. Also, these two theorems yield a polynomial algorithm to determine $\varphi(G)$ of an outerplanar graph with girth at least 8.

4.4 Graphs with large girth

As mentioned for regular graphs and seen in the case of outerplanar graphs, girth can play a significant role in computing $\varphi(G)$. The first result depending on girth and existence of a good set is from Campos et al., see [14].

Theorem 4.11 [14] *Let G be a graph with girth at least 9. If G has a good set, then $\varphi(G) = m(G)$.*

If the girth is getting smaller, then $\varphi(G)$ can be smaller than $m(G)$. This was observed by Campos et al. in [17], where it was also shown that it cannot be too far from $m(G)$ and is therefore a generalization of results on outerplanar graphs.

Theorem 4.12 [17] *If G is a graph of girth at least 7, then $\varphi(G) \geq m(G) - 1$.*

Moreover, the proof of above theorem yields a polynomial algorithm that produces an optimal b-coloring of G. In particular, the existence of a good set implies that $\varphi(G) = m(G)$. In the same paper they also posed the question, if girth can be lowered to 6 or even to 5. This question was answered for bipartite graphs with girth 6 by Kouider and Zamime in [59].

Theorem 4.13 [59] *Let $m \geq 3$ be an integer. Let G be a bipartite graph of girth at least 6 with bipartition (X, Y) such that $|X| = m$. If each vertex of X has degree $m - 1$ and each vertex of Y has degree at most $m - 2$, and each vertex has at most $\sqrt{m} - 1$ neighbors in common with the other vertices, then $\varphi(G) = m$.*

4.5 Kneser graphs

Let $S = \{1, \ldots, n\}$ and let V be the set of all k-subsets of S, where $k \leq \lfloor \frac{n}{2} \rfloor$. The *Kneser graph* $K(n, k)$ is the graph with vertex set V where two vertices are adjacent if and only if the corresponding subsets are disjoint. The following result from Javadi and Omoomi [48] has a straightforward proof with the construction of the appropriate b-coloring.

Theorem 4.14 [48] *For every integer $k \geq 3$ we have that $\varphi(K(2k + 1, k)) = k + 2$.*

The same paper contains also the following result. Its proof contains somewhat surprising connection with Steiner triple systems.

Theorem 4.15 [48] *For every positive integer n, $n \neq 8$, we have that*

$$\varphi(K(n, 2)) = \begin{cases} \left\lfloor \frac{n(n-1)}{6} \right\rfloor & ; \quad n \text{ is odd} \\ \left\lfloor \frac{(n-1)(n-2)}{6} \right\rfloor + 3 & ; \quad n \text{ is even} \end{cases} .$$

In [48] the authors conjecture that $\varphi(K(m,n)) = \Theta(m^n)$. Hajiabolhassan gives a positive answer to this question in [37] with the following result, since the upper bound follows directly from the degree of Kneser graph.

Theorem 4.16 [37] *Let $n \geq 3$ be an integer. If $m \geq 2n$, then*

$$\varphi(K(m,n)) \geq 2\binom{\lfloor \frac{m}{2} \rfloor}{n}.$$

The work on the b-chromatic number of Kneser graphs was continued by Balakrishnan and Kavaskar in [5] with the next upper bound.

Theorem 4.17 [5] *If $G = K(2n+k, n)$ is a Kneser graph of degree d with $|V(G)| \leq 2d + 2 - 2i$, where $i \geq 0$ and $n \geq 2$, then $\varphi(K(m,n)) \leq d - i$.*

For $i = 0$ in the above theorem, we get the following result which contains a condition on k.

Corollary 4.18 [5] *If $k > \left\lceil \frac{n}{2^{1/n}-1} - 1 \right\rceil$, then $\varphi(K(2n+k,n)) \leq d$, where d is the degree of $K(2n+k,n)$.*

5 Concepts connected with b-colorings

In this section we review the results on three concepts closely related with b-colorings: $m(G)$-tight graphs, b-critical graphs, b-perfect graphs and b-continuous graphs.

5.1 $m(G)$-tight graphs

A graph G is *m-tight* if it has exactly $m(G)$ vertices of degree $m(G)-1$. In contrast to the classic graph classes as presented in previous subsections, notice that tight graphs are defined in connection with $m(G)$, which is an upper bound of $\varphi(G)$. Tight graphs were first investigated by Linhares-Sales and Sampaio in [63]. The work was continued in [38] and [62]. There is a strong connection between the chromatic and the b-chromatic number for such graphs.

Definition 5.1 [63] *Let G be a m-tight graph. The b-closure of G, denoted G^*, is the graph with vertex set $V(G^*) = V(G)$, and edge set*

$$\begin{aligned} E(G^*) = E(G) &\cup \{uv \,|\, u \text{ and } v \text{ are non-adjacent dense vertices }\} \\ &\cup \{uv \,|\, u \text{ and } v \text{ are vertices with a common dense neighbor }\}. \end{aligned}$$

Lemma 5.2 [63] *Let G be a m-tight graph. Then $\varphi(G) = m(G)$ if and only if $\chi(G^*) = m(G)$.*

In the same article Linhares-Sales and Sampaio also present a generalization of pivoted trees.

Definition 5.3 [63] *Let G be an m-tight graph. We say that G is pivoted if there is a set N of non-dense vertices, with $|N| = k$, and a set of dense vertices D, with $|D| = m - k + 1$, satisfying:*

1. *For every pair $u, v \in N$, u is adjacent to v, or there is a dense vertex w that is adjacent to both u and v.*
2. *For every pair $u \in N$, $d \in D$, either u is adjacent to d or u and d are both adjacent to a dense vertex w (not necessarily in D).*

The next two results are a generalization of Theorem 4.2. The main tool used in the proof is the b-*closure* of a tight graph G, which is the graph G^*.

Theorem 5.4 [63] *If G is an m-tight graph. Then G is a pivoted graph if and only if $\omega(G^*) > m(G)$.*

Corollary 5.5 [63] *Let G be a m-tight graph. If G is a pivoted graph, then $\varphi(G) < m(G)$.*

Lin and Chang [62] connected tight graphs with Erdős-Faber-Lovász Conjecture. Denote by $\mathcal{K}_m$ the class of graphs $H = \bigcup_{i=1}^{m} K_m^i$, where K_m^i is a complete graph of m vertices for $1 \leq i \leq m$ and $|V(K_m^i) \cap V(K_m^j)| \leq 1$ for $i \neq j$.

Conjecture 5.6 (Erdős-Faber-Lovász) *If $H \in \mathcal{K}_m$, then $\chi(H) = m$.*

Let $\mathcal{B}_m$ denote the class of $m(G)$-tight bipartite graphs G, in which D and $D' = \bigcup_{x \in D} N_G(x)$ are stable sets and $|N_G(x) \cap N_G(x')| \leq 1$ for any two distinct dense vertices x and x'. In [62] the authors raised a new conjecture.

Conjecture 5.7 [62] *If $G \in \mathcal{B}_m$, then $\varphi(G) = m(G)$ or $m(G) - 1$.*

The next theorem proves that the Conjecture 5.7 is true for one special class of graphs, and hence shows, that it is weaker than the conjecture of Erdős-Faber-Lovász.

Theorem 5.8 [62] *If Erdős-Faber-Lovász Conjecture is true, then $\varphi(G) = m(G)$ or $m(G) - 1$ for any $G \in \mathcal{B}_m$.*

We end this subsection with a short discussion on $m(G)$-tight graphs with respect to their girth. Let X the set of vertices of degree $m-1$, and Y the complement. We say that $y \in Y$ 2-dominates X if $N_X(y) \cup N_X(N_X(y)) = X$. Kouider and Zamime extended Theorem 4.11 to $m(G)$-tight graphs of girth at least 8 and showed the following theorem.

Theorem 5.9 [59] *Let G be an $m(G)$-tight graph of girth at least 8. Then*

1. *$\varphi(G) = m(G)$ if and only if there is no vertex of Y which 2-dominates X.*
2. *$\varphi(G) = m(G) - 1$ if and only if there exists a vertex of Y which 2-dominates X.*

Theorem 5.9 has a nice corollary which includes the class $\mathcal{B}_m$ introduced in this subsection.

Corollary 5.10 [59] *Any graph $G \in \mathcal{B}_m$ of girth 8 has b-chromatic number $m(G)$.*

5.2 b-critical graphs

A graph G is *edge* b-*critical* if the removal of any edge of G decreases its b-chromatic number. Similarly, a graph G is *vertex* b-*critical* if the removal of any vertex of G decreases its b-chromatic number. The study of edge b-critical graphs began in 2010 by Ikhlef-Eschouf in [43]. Probably the most important result, which bounds the class of edge b-critical graphs, is as follows.

Theorem 5.11 [43] *If G is an edge b-critical graph, then $\varphi(G) = \Delta(G) + 1$.*

In [43] one can also find characterizations of edge b-critical graphs among P_4-sparse graphs and among quasi line graphs. The study of vertex b-critical graphs was initiated by Blidia et al. in [10]. They concentrated mainly on trees and succeeded to characterize all vertex b-critical trees provided that $\varphi(T) = \Delta(T)$ or $\Delta(T) + 1$ holds. They finished their work in [30] by characterizing the trees that are edge b-critical. The story is as follows.

Theorem 5.12 [10] *Let T be a vertex b-critical tree and let c be a b-coloring of T with $\varphi(T)$ colors. Let B be the set of all b-vertices of c. Then:*

1. *c does not have two b-vertices of the same color, i.e., $|B| = \varphi(T)$.*
2. *Every vertex u in $V(T) - B$ satisfies $d_T(u) \leq \varphi(T) - 1$.*
3. *Every vertex x in B satisfies $\varphi(T) - 1 \leq d_T(x) \leq \varphi(T)$.*

With the help of Theorem 5.12 one can prove the following corollary.

Corollary 5.13 [10] *If T is a vertex b-critical tree, then $\Delta(T) \leq \varphi(T) \leq \Delta(T) + 1$.*

Blidia et at. defined two classes of graphs, namely $\mathcal{T}_1$ and $\mathcal{T}_2$, which help to characterize all vertex and edge b-critical trees.

Definition 5.14 [10] *(Class $\mathcal{T}_1$). A tree T is in class $\mathcal{T}_1$ if for some integers k and p with $k \geq 4$ and $2 \leq p \leq k-2$, the vertex-set of T can be partitioned into four sets $\{v\}$, D_1, D_2, X with the following properties:*

1. *$|D_1| = p$, and every vertex of D_1 is adjacent to v;*

2. *$|D_2| = k-p$, and every vertex of D_2 has a neighbor in D_1;*

3. *every vertex of X has a neighbor in $D_1 \cup D_2$;*

4. *there is a vertex $w \in D_1$ such that w has a neighbor in D_2, w has degree k, and every vertex of $(D_1 \cup D_2) - \{v\}$ has degree $k-1$.*

Theorem 5.15 [10] *Let T be a tree with $\varphi(T) = \Delta(T)$. Then T is vertex b-critical if and only if $T \in \mathcal{T}_1$.*

Definition 5.16 [10]*(Class $\mathcal{T}_2$). A tree T is in $\mathcal{T}_2$ if there is a sequence $T_1, \ldots, T_k$ of trees, $k = \Delta(T)+1$, with $T = T_k$, where T_1 is a star of order k, and, for each $i \in \{1, \ldots, k-1\}$, T_{i+1} can be obtained from T_i by one of the listed operations:*

1. *Operation O_1: Add a new star of order $k-1$ and identify its center with one leaf of T_i whose neighbor has degree $k-1$.*

2. *Operation O_2: Add a new star of order $k-1$ with center x and add an edge between x and any vertex u of T_i such that $1 \leq d_{T_i}(u) \leq k-3$.*

3. *Operation O_3: Add a new star of order k and add an edge between one of its leafs and any vertex u of T_i such that $1 \leq d_{T_i}(u) \leq k-3$.*

Let $\mathcal{P}$ be the class of pivoted trees. The pivoted trees are not b-critical, which was proved in [30].

Theorem 5.17 [10] *Let T be a tree with $\varphi(T) = \Delta(T)+1$. Then T is vertex b-critical if and only if $T \in \mathcal{T}_2 - \mathcal{P}$.*

The general result for edge b-critical trees was proved by Eschouf et al.

Theorem 5.18 [30] *Let T be a tree. Then T is edge b-critical if and only if the following three conditions hold:*

- *$T \notin \mathcal{T}_1$;*
- *T has exactly $\Delta(T)+1$ vertices of degree $\Delta(T)$;*
- *the set of vertices of T of degree at most $\Delta(T)-1$ is a stable set.*

5.3 b-perfect graphs

A graph G is called b-perfect if for every induced subgraph H of G, we have $\varphi(H) = \chi(H)$. The main idea is to find and characterize all graphs which are b-perfect. Next result by Hoang and Kouider characterizes all bipartite graphs which are b-perfect.

Theorem 5.19 [40] *Let G be a bipartite graph. Then the following two conditions are equivalent:*

(i) G is b-perfect.

(ii) G is P_5-free, $3P_3$-free, and $(P_4 + P_3)$-free.

Later, Hoang et al. in [42] derived a list of 22 graphs $F_1, \ldots, F_{22}$, see Fig. 2. For $\mathcal{F} = \{F_1, \ldots, F_{22}\}$ they conjectured that G is b-berfect if and only if G does not contain any member of $\mathcal{F}$ as an induced subgraph (is $\mathcal{F}$-free for short). They have proved the conjecture for the following classes.

Theorem 5.20 [42] *A diamond-free graph is b-perfect if and only if it is $\{F_1, F_2, F_3, F_{18}, F_{20}\}$-free.*

Theorem 5.21 [42] *Let G be a graph with chromatic number at most 3. Then G is b-perfect if and only if it is $\mathcal{F}$-free.*

A characterization of b-perfect graphs among chordal graphs was presented by Maffray and Mechebbek in [64]. Some graphs of $\mathcal{F}$ can be removed, since they are not chordal as such.

Theorem 5.22 [64] *A chordal graph is b-perfect if and only if it is $\{F_1, \ldots, F_9\}$-free.*

The conjecture itself was confirmed by Hoang et al. in [41].

Theorem 5.23 [41] *A graph is b-perfect if and only if it is $\mathcal{F}$-free.*

Some variations of this theorem which use a combination of those 22 graphs were proved in the same article. In [51] Karthick and Maffray give a structural characterization of all claw-free b-perfect graphs beside the following forbiden subgraph characterization.

Theorem 5.24 [51] *A claw-free graph is b-perfect if and only if it is $\{F_1, F_2, F_3, F_4, F_6, F_{10}, F_{16}, F_{17}\}$-free.*

In [8] b-perfectness was defined with respect to the Grundy number $\Gamma(G)$. A graph G is b-Γ-perfect if $\varphi(H) = \Gamma(H)$ for every induced subgraph H of G. All such graphs have been characterized there as a subclass of P_4-free graphs.

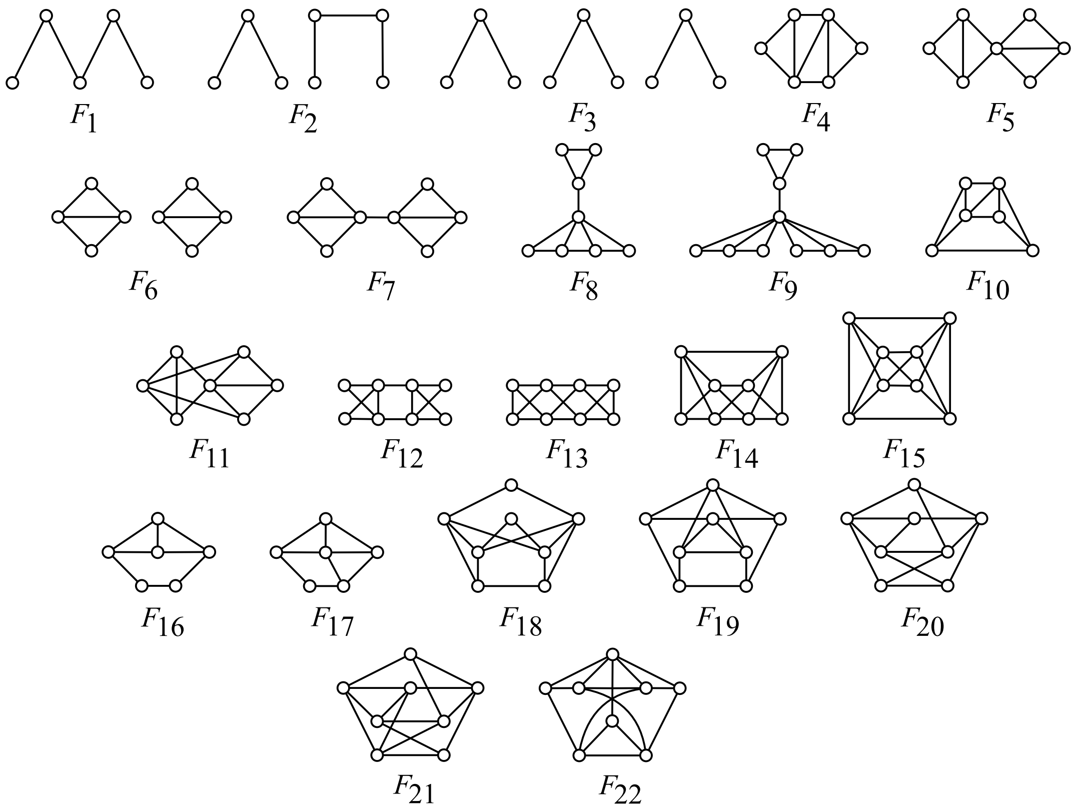

Figure 2: List of 22 forbiden subgraphs

5.4 b-continuous graphs

A graph G is said to be *b-continuous* if and only if for every integer k, $\chi(G) \leq k \leq \varphi(G)$, there exists a b-coloring with k colors. As it turns out not all graphs are b-continuous. Different approaches have been used to search for graphs that are b-continuous. In [1], Alkhateeb and Kohl analyzed disconnected graphs and a relation to the independence number, the minimum degree and the clique number. We mention only the most important results.

Theorem 5.25 [1] *Let G be a disconnected graph with components $G_1, \ldots, G_r$. If there exists a b-continuous component G_h, $1 \leq h \leq r$, such that $\varphi(G_h) \geq \max_{1 \leq i \leq r}\{\varphi(G_i)\} - 1$, then G is b-continuous.*

Theorem 5.26 [1] *Let G be a graph of order n. If $\alpha(G) = 2$ or $\delta(G) \geq n - 3$ or $\omega(G) \geq n - 4$ or $\alpha(G) \geq n - 2$, then G is b-continuous.*

As in Theorem 4.15 also the following result on Kneser graphs is proved via connection with Steiner triple systems.

Theorem 5.27 [48] *For every integer n, $n \geq 17$, Kneser graph $K(n, 2)$ is b-continuous.*

The b-spectrum $S_b(G)$ of a graph G is the set of all integers k for which there exists a b-coloring of G with k colors. In [5] there is an interesting result about b-spectrum of Kneser graphs.

Theorem 5.28 [5] *For any two positive integers n, k with $3 \leq n \leq k+1$, the b-spectrum of the Kneser graph $G = K(2n+k, n)$ contains the set $\{\chi(G) = k+2, k+3, \ldots, n+k+1\}$.*

A similar theorem was proved by Shaebani in [69]. Note, that Theorem 5.28 does not include Shaebani's theorem since it only works for cases where n is smaller then k.

Theorem 5.29 [69] *For each $k \in \mathbb{N}$, $K(2k+1, k)$ is b-continuous.*

Theorem 5.29 was proved with the help of semi-locally-surjective graph homomorphisms which are defined as follows. Let G and H be two graphs. The mapping f from $V(G)$ to $V(H)$ is called a *semi-locally-surjective graph homomorphism* whenever f is a surjective mapping and for each $y \in V(H)$, there exists $x \in f^{-1}(u)$ such that $f(N_G(x)) = N_H(y)$.

It is a natural question to ask if for a given set of integers I there exists a graph G whose spectrum equals I. This question was answered by Barth et al. in [6].

Theorem 5.30 [6] *For any finite nonempty set $I \subset (\mathbb{N} - \{1\})$ there exists a graph G such that $S_b(G) = I$.*

In the section of regular graphs the relation between b-coloring and girth is presented in the sense that every regular graph G which has girth large enough has a b-coloring with $\Delta(G)+1$ colors. The size of the girth of a graph is also important in proving that a graph is b-continuous. This result was proved in [5].

Theorem 5.31 [5] *If G is k-regular graph with girth at least 6 having no cycles of length 7, then G is b-continuous.*

In [69], Shaebani deals with regular graphs with small degree and again uses semi-locally-surjective graph homomorphisms, and the Theorem 3.11 by Blidia et. al. [11], to prove the following result.

Theorem 5.32 [69] *Let $3 \leq d \leq 6$ and for each $2 \leq i \leq d$, G_i be an i-regular graph with girth $g(G_i) \geq 5$ which is different from the Petersen graph. Also, suppose that for each $3 \leq i \leq d$, there exists a semi-locally-surjective graph homomorphism f_i from G_i to G_{i-1}. Then for each $2 \leq i \leq d$, G_i is b-continuous.*

Several other cases were considered for the b-continuity. For instance, we mention the results on P_4-tidy graphs from [7] and P_4-sparse graphs and cographs from [12]. Let G be a graph and A a P_4 in G. A *partner* of A is a vertex v in $G - A$ such that $A \cup \{v\}$ induces at least two P_4's in G. A graph G is *P_4-sparse* if no induced P_4 has a partner and *P_4-tidy* if every induced P_4 has at most one partner.

Theorem 5.33 [7, 12] *If G is P_4-tidy or P_4-sparse or a cograph, then G is b-continuous.*

6 Graph operations

In this section we overview the results on the b-chromatic number with respect to graph operations. We will omit the definitions of operations (they can be found in the original works) and majority of results on special families of graphs in order not to exceed the length of this survey. We cover graph powers, the Cartesian product, the strong product, the lexicographic product and the direct product in the following subsections. Note also that some special results on the corona product can be found in [73].

6.1 Graph powers

The powers of paths and of cycles where considered by Effantin and Kheddouci in [23].

Theorem 6.1 [23] *For two integers $p, n \geq 1$ we have that*

$$\varphi(P_n^p) = \begin{cases} n & ; \quad n \leq p+1 \\ p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor & ; \quad p+2 \leq n \leq 4p+1 \\ 2p+1 & ; \quad n \geq 4p+2 \end{cases} .$$

Theorem 6.2 [23] *For integers $p \geq 1$ and $n \geq 3$ we have that*

$$\varphi(C_n^p) = \begin{cases} n & ; \quad n \leq 2p+1 \\ p+1 & ; \quad n = 2p+2 \\ p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor & : \quad 3p+2 \leq n \leq 4p \\ 2p+1 & ; \quad n \geq 4p+1 \end{cases} .$$

Moreover, if $2p+3 \leq n \leq 3p$ we have that

$$\varphi(C_n^p) \geq \min\left\{ n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor \right\} .$$

The last result was later improved by Kohl [53].

Theorem 6.3 [53] *Let C_n^r be the r-th power of a cycle of order n with $2r+3 \leq n \leq 3r$. Moreover, set $\ell = n - 2(r+1)$, $R = (r+1+3\ell) \pmod{5\ell}$ and $m = \max\left\{\left\lfloor \frac{R-2\ell}{3} \right\rfloor, 0\right\}$. Then, $\varphi(C_n^r) = \frac{3n-R}{5} + m$.*

In [53] it was observed that a conjecture on the b-chromatic number of powers of cycles from [23] does not hold in general. In between, powers of complete caterpillars and of k-ary trees where considered in [22] and [24], respectively. The methods used are similar to those in [23].

6.2 The Cartesian product

The general lower bound for the b-chromatic number of the Cartesian product was presented by Kouider and Maheo in [55] and the upper bound followed by the same authors in [56].

Theorem 6.4 [55] *For any two graphs G and H, $\varphi(G \,\Box\, H) \geq \max\{\varphi(G), \varphi(H)\}$.*

Theorem 6.5 [56] *Let G be a graph with girth at least 7 and let H be a graph of order n_H. Then*

$$\varphi(G \,\Box\, H) \leq \varphi(G)(n_H + 1) + \Delta(H) + 1.$$

The lower bound can be improved under certain additional conditions as shown in [55].

Theorem 6.6 [55] *Let G and H be two graphs such that both G and H have a stable set of b-vertices of all colors. Then $\varphi(G \,\Box\, H) \geq \varphi(G) + \varphi(H) - 1$ and the graph $G \,\Box\, H$ has a stable set of b-vertices of all colors.*

This result was slightly improved in [3], mainly with respect to avoid the condition of the stable set of b-vertices of all colors.

In [55], the exact values for $\varphi(K_{1,n} \,\Box\, K_{1,n})$ and for $\varphi(K_{1,n} \,\Box\, P_k)$ have been presented as well as an upper bound for $\varphi(K_n \,\Box\, K_p)$. Further, exact results on $\varphi(K_m \,\Box\, C_n)$ and $\varphi(K_m \,\Box\, P_n)$ were derived in [49], where also the upper bound for $\varphi(K_n \,\Box\, K_n)$ was established.

The deepest results on the b-chromatic number of the Cartesian product are presented by Maffray and Silva in [66]. They use similar methods as the ones for trees in the pilot paper from Irving and Manlove [44] using dense edges, encircled vertices and good sets.

Theorem 6.7 [66] *If $n \geq 4, k \geq 5, p \geq 2$ are integers and T is a tree, then*

1. *$\varphi(T \,\Box\, C_n) = m(T \,\Box\, C_n)$.*
2. *if $H = T \,\Box\, P_k$ has a good set, then $\varphi(H) = m(H)$, otherwise $\varphi(H) = m(H) - 1$.*

3. $\varphi(T \square K_{1,p}) \geq m(T \square K_{1,p}) - 1$.

4. if $H = T \square K_{1,p}$ *has a good set, then* $\varphi(H) = m(H)$.

5. if $H = T \square K_{1,p}$ *does not have a good set and* $m(H) \geq 4$*, then* $\varphi(H) = m(H) - 1$.

In addition, let us mention that some bounds on the generalized Hamming graphs are proven in [19].

6.3 The strong product

As seen above the b-chromatic number was first studied for the Cartesian product of graphs, which appears no easier then studying the factors separately. A natural way was to study other products. A lot of interesting results were given by Jakovac and Peterin in [46]. First, they show that the b-chromatic number of the strong product of arbitrary graphs is a sub-multiplicative function.

Theorem 6.8 [46] *Let* G *and* H *be arbitrary graphs. Then*

$$\varphi(G \boxtimes H) \geq \varphi(G)\varphi(H)\,.$$

From the previous results a lot of graphs have the property that their b-chromatic number equals the trivial upper bound in respect of their maximum degree. Therefore, the following result yields a large family of graphs G and H for which $\varphi(G \boxtimes H) = \varphi(G)\varphi(H)$.

Corollary 6.9 [46] *If* $\varphi(G) = \Delta(G) + 1$ *and* $\varphi(H) = \Delta(H) + 1$*, then*

$$\varphi(G \boxtimes H) = \varphi(G)\varphi(H).$$

In the same paper two exact results where given where one of the factors is a path P_n or a cycle C_n.

Theorem 6.10 [46] *Let* H *be an arbitrary graph. Then* $\varphi(P_n \boxtimes H) = 3(\Delta(H)+1) = \varphi(C_n \boxtimes H)$ *for any* $n \geq 3(\Delta(H) + 1) + 2$.

This theorem was a basis for several exact results on the b-chromatic number of the strong product of some special graphs as paths, cycles and stars. In particular, $\varphi(P_k \boxtimes K_{1,k-2}) = k$ shows that $\varphi(G \boxtimes H)$ is not bounded from above by any function of $\varphi(G)$ and $\varphi(H)$.

6.4 The lexicographic product

The results on the strong product of graphs were extended to the lexicographic product where it was also shown that the b-chromatic number of the lexicographic product of arbitrary graphs is a sub-multiplicative function.

Proposition 6.11 [46] *Let G and H be arbitrary graphs. Then*

$$\varphi(G[H]) \geq \varphi(G)\varphi(H)\,.$$

Similarly, exact results were determined in the case where only one of the factors is fixed.

Theorem 6.12 [46] *Let H be an arbitrary graph. Then $\varphi(P_n[H]) = 2|V(H)| + \Delta(H) + 1 = \varphi(C_n[H])$, for each $n \geq 2\,(2|V(H)| + \Delta(H) + 1) + 1$.*

Theorem 6.13 [46] *Let H be a graph and $k, \ell \geq \Delta(H) + 1$. Then $\varphi(K_{k,\ell}[H]) = 2(\Delta(H) + 1)$.*

The later two results yield exact values for $\varphi(P_k[P_\ell])$, $k \geq 5$ and $\ell \geq 2\left\lceil \frac{2k}{3} \right\rceil + 3$, and $\varphi(K_{s,t}[K_{\ell,k}])$. In particular, $\varphi(K_{k,k}[K_{k,k}]) = 2k$ shows that $\varphi(G[H])$ is not bounded from above by any function of $\varphi(G)$ and $\varphi(H)$.

6.5 The direct product

Some of the results on the strong and lexicographic product were also applied to the direct product of graphs. We only mention two of them.

Proposition 6.14 [46] *Let G and H be two arbitrary graphs. Then*

$$\varphi(G \times H) \geq \max\{\varphi(G), \varphi(H)\}\,.$$

Theorem 6.15 [46] *Let H be a graph with at least one edge. Then $\varphi(P_n \times H) = 2\Delta(H) + 1$ for every $n \geq 2\,(2\Delta(H) + 1) + 1$.*

Again, the example $\varphi(P_{4k+3} \times K_{1,k}) = 2k+1$ shows that $\varphi(G \times H)$ is not bounded from above by any function of $\varphi(G)$ and $\varphi(H)$.

7 The b-chromatic index

A b-*edge coloring* of a graph G is a proper edge coloring of G such that each color class contains an edge that has at least one incident edge in every other color class. The b-*chromatic index of a graph* G is the largest integer $\varphi'(G)$ for which G has a b-edge coloring with $\varphi'(G)$ colors. An edge e of color i that has all other colors on its incident edges is called a b-*edge*. The upper bound $m'(G)$ for $\varphi'(G)$ is defined similarly as for the vertex version: $m'(G) = \max\{i \,|\, d(e_i) \geq i-1\}$, where $d(e_1) \geq \ldots \geq d(e_m)$ is the edge degree sequence of the edges $e_1, \ldots, e_m$ of G.

For the b-chromatic number of an arbitrary tree T there are only two possibilities. Namely, $\varphi(T) = m(T)$ or $\varphi(T) = m(T) - 1$. An analogous result was shown for the b-chromatic index of trees in [47]. However, this proof had a leak as shown by Silva in [72]. Maffray and Silva showed in [65] that the difference $m(G) - \varphi(G)$ can be arbitrarily large for a claw-free block graph G which is precisely the line-graph of a tree. Hence, $m'(T) - \varphi'(T)$ can be arbitrarily large for a tree T. Another example was later given by Silva in [72]. However, the difference can be small for some special classes of trees. In [16] the authors show the following theorem.

Theorem 7.1 [16] *If G is the line graph of a caterpillar, then $\varphi(G)$ is either $m(G)$ or $m(G) - 1$, and deciding its value can be done in polynomial time.*

Theorem 7.1 is equivalent to determining the b-chromatic index of caterpillars. Moreover, there always exists a b-edge coloring of caterpillars with only $m'(G) - 1$ colors [16].

Theorem 7.2 [16] *If G is the line graph of a caterpillar, then a b-coloring of G with $m(G) - 1$ colors exists and can be found in polynomial time.*

Since determining the b-chromatic index of trees is much harder then determining the b-chromatic number of trees one would expect that this problem was also NP-complete. Surprisingly, Campos and Silva proved in [18] that this is not the case. Again, this is done for the b-chromatic number of claw-free block graphs which is the same as proving this theorem for the b-chromatic index of trees.

Theorem 7.3 [18] *Given a claw-free block graph G and an integer $k > \omega$, where $\omega = \omega(G)$, it can be decided whether $\varphi(G) \geq k$ in time $O(\omega^4 k^3 n)$.*

Next result from [47] shows (not unexpected) that, roughly speaking, if the diameter is large enough in regular graphs, then $\varphi'(G) = m'(G)$ follows. For this we define the graph $G(v)$ as the subgraph of G induced by $N(v) \cup S_2(v)$ where $S_2(v)$ contains all vertices at distance two to v.

Theorem 7.4 [47] *Let G be an r-regular graph with* $\operatorname{diam}(G) \geq 4$ *and let u and v be two vertices at distance at least 4. If $G(u)$ and $G(v)$ are class 1 graphs with $\Delta(G(u)) = \Delta(G(v)) = r - 1$, then*

$$\varphi'(G) = 2r - 1.$$

One can expect a generalization of Theorem 7.4 to non-regular case and its use for many classes of graphs. In particular, the following two theorems from Koch and Peterin [52] are a consequence of the above theorem and a certain properties of the direct product. Namely, there are three sufficient conditions in Theorem 7.4 and these conditions are used for the direct product. This is done in [52] with three lemmas. Everything is summarized in the next two theorems.

Theorem 7.5 [52] *Let G and H be connected graphs, r_G- and r_H-regular, respectively, and without triangles. Let additionally $gg' \in E(G)$, where g and g' do not lie on any five cycle and $\Delta(G(g)) = r_G - 1 = \Delta(G(g'))$. If at least one of G and H is nonbipartite, then*

$$\varphi'(G \times H) = 2r_G r_H - 1.$$

Theorem 7.6 [52] *Let G and H be connected bipartite graphs, and r_G- and r_H-regular, respectively. Let* $\mathrm{diam}(G) \geq 4$. *If there exists $g, g' \in V(G)$ with $d_G(g, g') \geq 4$ and $\Delta(G(g)) = r_G - 1 = \Delta(G(g'))$, then*

$$\varphi'(G \times H) = 2r_G r_H - 1.$$

In contrast to the vertex case, it is not hard to find a b-edge coloring with $m'(G)$ colors of regular graphs of large enough girth.

Theorem 7.7 [47] *If G is a d-regular graph with girth $g \geq 5$, then*

$$\varphi'(G) = 2d - 1.$$

While the above results show that in sparse graphs we can expect the equality $\varphi'(G) = 2d - 1$, one can expect certain problems to find the b-chromatic index for dense graphs. The only tool for this until now is the following result. For this note that a graph G is *d-edge regular* if its line graph is d-regular.

Theorem 7.8 [47] *If G is a d-edge regular graph with minimum degree $\delta \geq 4$ and $\varphi'(G) = d + 1$, then at most two edges of any four cycle and at most two edges of any triangle can be b-edges in a $(d + 1)$-b-edge coloring of G.*

With the help of this it was shown in [47] that if G is a complete k-partite graph $K_{n,\ldots,n}$, $1 \leq n$, or a complete bipartite graph $K_{s,t}$, then $\varphi'(G) < m'(G)$. However, this is still far from the exact result.

The next lemma is proven by construction of a partial b-edge coloring with $m'(G)$ colors and then extended to the whole graph and forms a basis for the proof of Theorem 7.10, which is an edge analogue of Theorem 3.3 with different four exceptions. The proof is more elegant because of this lemma.

Lemma 7.9 [47] *Let G be a 3-regular graph. If G has an induced cycle C_5 or an induced path P_6, then $\varphi'(G) = 5$.*

Theorem 7.10 [47] *Let G be a connected cubic graph. Then $\varphi'(G) = 5$ if and only if G is not isomorphic to K_4, $K_3 \,\square\, K_2$, $K_{3,3}$ or 3-cube.*

We end this section with the following two results.

Theorem 7.11 [52] *Let G be a graph and H an r-regular graph. If H is a class 1 graph, then*

$$\varphi'(G \times H) \geq r\varphi'(G).$$

The above lower bound is better when r is small and for $r = 2$ we have that $H \cong C_{2n}$ and we get

$$2d + 2 \leq \varphi'(G \times C_{2n}) \leq 2d + 3 = m'(G \times C_{2n})$$

for any d-edge regular graph G with $\varphi'(G) = d + 1$, as observed in [52]. Recall that the condition $\varphi'(G) = d + 1$ is not true only for finite number of d-edge regular graphs by Theorem 3.1.

The reason for better performance of this bound for small r is hidden in the construction of a b-coloring. Namely, every edge of a direct product projects to edges in both factors and the coloring presented there is a combination of both colors. Hence every edge in a product has two incident edges of the same color (one in each end-vertex) which can be far from optimal for big r.

A generalization of Theorem 7.11 from regular graph H to arbitrary graphs can be done by the following concept. Let H be a proper edge colored graph. If there exist an edge in every color class for which both endvertices have all colors on edges incident to it, then we call such a coloring an *edge symmetric coloring* of H. Unfortunately, edge symmetric colorings do not exist for class 2 graphs as well as for many class 1 graphs, since we need for such a coloring at least $\Delta(H)$ edges in which both endvertices have degree $\Delta(H)$. On the other hand, one does not need the regularity anymore.

Theorem 7.12 [52] *Let G and H be graphs. If H contains an edge symmetric coloring, then*

$$\varphi'(G \times H) \geq \varphi'(G)\Delta(H).$$

8 Complexity and algorithmic aspects

The definition of b-coloring immediately implies an easy algorithm for finding a b-coloring of a graph G. For this, one needs to color vertices of G by any proper coloring, say with k colors. If not otherwise, the greedy algorithm is fine, i.e. until all vertices are not colored choose any non-colored vertex and assign to it the smallest color which does not exist within its neighborhood. Such a coloring is not always a b-coloring since some color classes can be without a b-vertex. However, one can easily check every color class for its b-vertex. If such a vertex does not exist, then we simply recolor all the vertices of this color class to obtain a $(k-1)$-coloring of G. Clearly, we end-up with a b-coloring after all the original color classes from $\{1,\ldots,k\}$ have been checked for their b-vertices and recolored when needed. Such an approach was demonstrated by Elghazel et al. in [25]. While such a procedure produces a b-coloring of a graph, the number of colors may still be very far from the b-chromatic number.

In the same paper [25] b-colorings were used to analyze the data of hospital stays in French medical system. Another application of b-colorings is due to the postal mail sorting system, see [34, 35]. The method proposed was 98% successful.

8.1 Complexity

In this part we overview the results on complexity of problems related to b-colorings. The most classic one, discussed already in paper of Irving and Manlove [44], is certainly the following.

b-CHROMATIC NUMBER
Instance: A graph G and a positive integer k.
Question: Is $\varphi(G) \geq k$?

By the definition of the b-chromatic number one must expect the following result. It was obtained with the reduction from the NP-complete problem EXACT COVER BY 3-SETS.

Theorem 8.1 [44] *b-CHROMATIC NUMBER is NP-complete.*

Next problem from [60] is dealing with arbitrary b-colorings not only with those with maximum number of colors.

b-COLORING
Instance: A graph G and a positive integer k.
Question: Is there a b-coloring of G by k colors?

Surprisingly, b-COLORING is NP-complete even for bipartite graphs, which also yields that b-CHROMATIC NUMBER is NP-complete for bipartite graphs. The reduction was done from the NP-complete problem 3-EDGE COLORABILITY OF 3-REGULAR GRAPHS.

Theorem 8.2 [60] *b-COLORING is NP-complete for $k = m(G)$ even for connected bipartite graphs and $m(G) = \Delta(G) + 1$.*

Theorem 8.2 also holds for chordal graphs [38] and line graphs [16]. In fact, the result on chordal graphs is even stronger. Recall that an $m(G)$-tight graph G has exactly $m(G)$ vertices of degree $m(G)-1$. The next problem is from [38].

TIGHT b-CHROMATIC PROBLEM
Instance: An $m(G)$-tight graph G.
Question: Is $\varphi(G)$ equal to $m(G)$?

Theorem 8.3 [38] *The TIGHT b-CHROMATIC PROBLEM is NP-complete for connected bipartite graphs and for connected chordal distance-hereditary graphs.*

The reduction was done from 3-EDGE COLORABILITY for connected bipartite graphs and from 3-EDGE COLORABILITY OF 3-REGULAR GRAPHS for connected chordal distance-hereditary graphs. Conversely, it was shown in [38] that the TIGHT b-CHROMATIC PROBLEM is polynomial-solvable for trees, complements of bipartite graphs, P_4-sparse graphs, split graphs and block graphs.

We continue with the problem on b-continuity from [6].

b-CONTINUOITY
Instance: A graph G, b-colorings with $\chi(G)$ and $\varphi(G)$ colors.
Question: Is G b-continuous?

The latest problem is also NP-complete and the reduction is again done from EXACT COVER BY 3-SETS problem.

Theorem 8.4 [6] *The problem b-CONTINUOITY is NP-complete.*

The b-CONTINUITY problem remains NP-complete even restricted to chordal graphs as shown independently by Faik [31] and Kára et al. in [50] and to bipartite graphs shown by Faik in [32].

Deciding whether $\varphi(G) \geq k$ for a fixed integer k is Fixed Parameter Tractable (FPT for short) is still open for an arbitrary graph G. However, Silva presented an algorithm for block graphs in her thesis [71] which is FPT. In [39] Havet and Sampaio introduced DUAL OF b-COLORING problem and showed that it is Fixed Parameter Tractable. For more details see [39].

We end this subsection with the complexity for the edge version of the b-chromatic number, i.e. the b-chromatic index. The problem whether $\varphi'(G) = m'(G)$ was shown to be NP-complete by Campos et al. [16, 61].

EDGE b-COLORING
Instance: A graph G.
Question: Is $\varphi'(G)$ equal to $m'(G)$?

Theorem 8.5 [16, 61] *EDGE b-COLORING is NP-complete, even if G is either a comparability graph or a C_k-free graph, for $k \geq 4$.*

The result was obtained with the reduction from the NP-complete problem of EDGE-COLORABILITY, which poses the question if for a given integer $k \geq 3$ graph G can be properly edge colored with k colors.

8.2 Approximation results

The first approximation approach on the b-chromatic number of a graph G was done by Corteel et al. in [20]. They establish a connection between MAX 3-ESAT problem (see [20] and the references therein for the definition) and the b-chromatic number. The following result is obtained based on known approximation results for the MAX 3-ESAT problem.

Theorem 8.6 [20] *The b-chromatic number problem is not approximable within $120/113 - \epsilon$ for any $\epsilon > 0$, unless $P = NP$.*

They also posed a question about the existence of a constant-factor approximation for the b-chromatic number. This was settled in negative by Galčik and Katrenič via the MAXIMUM INDEPENDANT SET.

Theorem 8.7 [36] *For all $\epsilon > 0$, it is NP-hard to approximate the b-coloring problem for graphs with n vertices within a factor $n^{1/4-\epsilon}$.*

Theorem 2.13 has two consequences regarding approximation algorithms.

Corollary 8.8 [57] *Let G be a connected d-regular bipartite graph with partition $V(G) = A \cup B$ with $|A| = |B| = n$ and $d < n$. If $n - d$ is equal to a constant $c \geq 1$, then there is a c-approximation algorithm for b-coloring of G with the maximum number of colors.*

Corollary 8.9 [57] *Let G be a connected bipartite graph with partition $V(G) = A \cup B$ with $|A| = |B| = n$. If $\Delta(G) < n$ and $n - \delta(G)$ equal to a constant $c \geq 1$, then there is a c-approximation algorithm for b-coloring of G with the maximum number of colors.*

8.3 Hybrid evolutionary algorithm

Fister et al. published the first heuristic algorithm for the b-chromatic number in [33]. Its base is an evolutionary algorithm which is crossed with some local properties. In this evolution algorithm one obtains new objects in a population (of some proper colorings of a graph) with a Greedy Partition Crossover from two parents (two colorings of the same graph) already existing in population or by a mutation of already existing object in the population. After repeating this step for a few times, one usually gets a coloring which is a b-coloring. Among all such colorings one chooses the one with the most colors, which is then the lower bound for the b-chromatic number. This procedure is then repeated several times to get better results. The algorithm was tested on d-regular graphs, $d \in \{3, 4, 5, 6, 7\}$ up to 12 vertices and on some special graphs which turn out to be problematic for the chromatic number. To test the obtained results, the authors also checked all mentioned regular graphs with the brute force algorithm. All obtained results on regular graphs were exact. The side effect of the brute force testing was the surprisingly big number of d-regular graphs with the property $\varphi(G) < d + 1$. Namely, there are only four such graphs among 3-regular graphs, see Theorem 3.3, but at least 462 such graphs among 4-regular graphs, at least 7276 such graphs among 5-regular graphs, at least 8128 such graphs among 6-regular graphs and at least 1533 such graphs among 7-regular graphs.

8.4 Linear programming

An integer linear programming model for computing $\varphi'(G)$ was introduced in [52]. It is based on the standard formulation of the vertex coloring problem translated to the edge version. Since this is an NP-hard problem, one cannot expect that the solutions obtained by this method always lead to exact values of $\varphi'(G)$ within reasonable time bounds. However, each solution represents a lower bound for $\varphi'(G)$. With the help of it, the solutions for $\varphi'(P_m \times P_n)$ and $\varphi'(C_m \times C_n)$ have been presented in [52]. The linear program was used to derive the values for small m and n, while for big enough m and n one can use Theorem 7.4.

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