On the b-chromatic number of some graph products

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Abstract

A b-coloring is a proper vertex coloring of a graph such that each color class contains a vertex that has a neighbor in all other color classes and the b-chromatic number is the largest integer $\varphi(G)$ for which a graph has a b-coloring with $\varphi(G)$ colors. We determine some upper and lower bounds for the b-chromatic number of the strong product $G \boxtimes H$, the lexicographic product $G[H]$ and the direct product $G \times H$ and give some exact values for products of paths, cycles, stars, and complete bipartite graphs. We also show that the b-chromatic number of $P_n \boxtimes H$, $C_n \boxtimes H$, $P_n[H]$, $C_n[H]$, and $K_{m,n}[H]$ can be determined for an arbitrary graph H, when integers m and n are large enough.

Key words: b-chromatic number; Strong product; Lexicographic product; Direct product.

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1 Introduction and preliminaries

A *b-coloring* of a graph G is a proper vertex coloring of G such that each color class contains a vertex that has at least one neighbor in every other color class and *b-chromatic number of a graph* G is the largest integer $\varphi(G)$ for which G has a b-coloring with $\varphi(G)$ colors. A vertex of color i that has all other colors in its neighborhood is called *color i dominating vertex*. The invariant $\varphi(G)$ has the chromatic number $\chi(G)$ as a trivial lower bound, however the difference between both of them can be arbitrary large [3]. A trivial upper bound for $\varphi(G)$ is $\Delta(G)+1$. Let $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$ be the degree sequence of G. Then $m(G) = \max\{i \,|\, d(v_i) \geq i-1\}$ is an improved upper bound for $\varphi(G)$, see [13].

The concept of the b-chromatic number was introduced by Irving and Manlove [13]. Since then the b-chromatic number has drawn much attention in scientific area, see [11, 15, 16, 20]. We can easily imagine the color classes as different communities, where every community i has a representative (the color i dominating vertex) that is able to communicate with all the others communities.

Even though the b-chromatic number is a simple concept, it is hard to determine the exact values, even for known families of graphs. This leads to studies of lower and upper bounds, see for instance [1, 2, 17, 19]. Determining the b-chromatic number in general is an NP-hard problem but Irving and Manlove proved that it is polynomial for trees [13]. The approximation for the b-chromatic number is treated in [7].

Regular graphs play an interesting role for the b-chromatic number since for every regular graph G equality $m(G) = \Delta(G)+1$ holds. It was proved in [16] and [20] that $\varphi(G) = \Delta(G)+1$ for a d-regular graph G on at least d^4 vertices. It was shown that this bound can be lowered to $2d^3$, see [5]. Also, in [14] it was proved that there are only four exceptions for cubic graphs with $\varphi(G) < \Delta(G)+1$, one of the exceptions being the Petersen graph. All the exceptions have no more then 10 vertices.

The b-chromatic number was also studied on powers of graphs [8, 9, 10], as it was studied for the Cartesian product of graphs [6, 17, 18], which appear no easier then studying the factors separately. Some bounds are determined with respect to the girth and the size of the factors. Exact values were determined for Cartesian products of paths, cycles, stars, complete graphs and hypercubes and it was shown that the b-chromatic number of the Cartesian product is not necessarily bounded by the b-chromatic number of its factors.

We continue the above work on other three standard products: the strong, the lexicographic, and the direct product. We give strict lower and upper bounds for them. The lower bounds can be derived from the b-chromatic number of factors of the product, while the upper bounds follow from the trivial upper bound. In fact we show that there is no upper bound with respect to the b-chromatic number of the factors for the strong, the lexicographic, and the direct product. We show similar results for $m(G)$. Along the way we derive some exact values for some special factors.

We end this section with the definition of strong, lexicographic, and direct products. For all three products of graphs G and H the vertex set of the product is $V(G) \times V(H)$. Their edge sets are defined as follows. Two vertices are adjacent in the *strong product* $G \boxtimes H$ if they are adjacent in both coordinates or if they are adjacent in one coordinate and equal in the other. In the *direct product* $G \times H$

two vertices are adjacent if they are adjacent in both coordinates. Finally, in the *lexicographic product* $G[H]$ two vertices are adjacent if they are adjacent in the first coordinate or they equal in the first and are adjacent in the second coordinate. Note that $G \times H$ is a spanning subgraph of $G \boxtimes H$ which is in turn a spanning subgraph of $G[H]$. Note also that the lexicographic product is not commutative.

For $v \in V(H)$ the subgraph induced with vertices $G^v = \{(u,v)|u \in V(G)\}$ is called a *G-fiber* of the product under consideration. Analogue we define H-fibers H^u. Note that G- and H-fibers of strong and lexicographic products are isomorphic to G and H, respectively, while in the case of the direct product fibers have no edges. For more on products we recommend the book [12].

2 Strong Product

In this section we determine some bounds for the strong product, give some exact values and also determine the b-chromatic number for products of some known family of graphs. In the end we conjecture an upper bound which happens to hold for all the studied cases bellow.

Theorem 2.1 *Let G and H be arbitrary graphs. Then*

$$\varphi(G \boxtimes H) \geq \varphi(G)\varphi(H)\,.$$

Proof. Let c_1 be a $\varphi(G)$-b-coloring of graph G and c_2 be a $\varphi(H)$-b-coloring of graph H. Let (u,v) be a vertex in $G \boxtimes H$. Define coloring c_3 of $G \boxtimes H$ as

$$c_3((u,v)) = (c_1(u), c_2(v))\,.$$

Coloring c_3 is obviously a proper vertex coloring of $G \boxtimes H$ with $\varphi(G)\varphi(H)$ colors. It remains to be seen that c_3 is a b-coloring. Let u be a color i dominating vertex in G and v a color j dominating vertex of H. Then vertex $(u,v) \in V(G \boxtimes H)$ is colored with color (i,j). Since u is adjacent to at least one vertex x_k in G colored k, $k \in \{1,2,\ldots,\varphi(G)\}\backslash\{i\}$ and v is adjacent to at least one vertex y_ℓ in H colored ℓ, $\ell \in \{1,2,\ldots,\varphi(G)\}\backslash\{j\}$, (u,v) is also adjacent to at least one vertex colored (k,ℓ), namely vertex (x_k,y_ℓ), $k \neq i$ and $\ell \neq j$. Furthermore (u,v) is adjacent to a vertex (u,y_ℓ) of color (i,ℓ) and a vertex (x_k,v) of color (k,j). Thus c_3 is a b-coloring and $G \boxtimes H$ can be b-colored with at least $\varphi(G)\varphi(H)$ colors. □

For graphs G and H, $\varphi(G \boxtimes H) \leq \Delta(G \boxtimes H) + 1$ and since $\Delta(G \boxtimes H) = \Delta(G)\Delta(H) + \Delta(G) + \Delta(H)$ we have

$$\varphi(G \boxtimes H) \leq \Delta(G)\Delta(H) + \Delta(G) + \Delta(H) + 1\,. \tag{1}$$

Inequality (1) lead us to the next corollary that also implies that the above upper and lower bounds are equal in many cases.

Corollary 2.2 *If $\varphi(G) = \Delta(G) + 1$ and $\varphi(H) = \Delta(H) + 1$, then*

$$\varphi(G \boxtimes H) = \varphi(G)\varphi(H)\,.$$

Proof. Let $\varphi(G) = \Delta(G) + 1$ and $\varphi(H) = \Delta(H) + 1$. Then

$$\varphi(G)\varphi(H) = \Delta(G)\Delta(H) + \Delta(G) + \Delta(H) + 1$$

which is by inequality (1) also the upper bound of $\varphi(G \boxtimes H)$. □

The contrapositive statement of Corollary 2.2 does not necessarily hold. Take for example graph $P_5 \boxtimes P_3$. Then $\varphi(P_5 \boxtimes P_3) = 6 = 3 \cdot 2 = \varphi(P_5)\varphi(P_3)$, but $\varphi(P_3) = 2 \neq 3 = \Delta(P_3) + 1$.

Next we concentrate on some exact results. By inequality (1) we have $\varphi(P_n \boxtimes H) \leq 3(\Delta(H) + 1)$ for $n \geq 3$. We can prove that this bound is sharp whenever n is sufficiently large (with respect to $\Delta(H)$).

Theorem 2.3 *Let H be an arbitrary graph. Then $\varphi(P_n \boxtimes H) = 3(\Delta(H) + 1) = \varphi(C_n \boxtimes H)$ for any $n \geq 3(\Delta(H) + 1) + 2$.*

Proof. Let $P_n = v_1 v_2 \ldots v_n$ and $n \geq 3(\Delta(H)+1)+2$. Let u be a vertex of maximum degree in H and u_i a vertex of $P_n \boxtimes H$ in the intersection of P_n^u- and H^{v_i}-fibers. Define three sets of colors. Let $C_1 = \{1, 2, \ldots, \Delta(H)+1\}$, $C_2 = \{\Delta(H)+2, \Delta(H)+3, \ldots, 2\Delta(H)+2\}$, and $C_3 = \{2\Delta(H)+3, 2\Delta(H)+4, \ldots, 3\Delta(H)+3\}$. Color vertex u_{3i-1}, $i \in \{1, 2, \ldots, \Delta(H) + 1\}$, with color i and all its neighbors in fiber $H^{u_{3i-1}}$ with the remaining colors of C_1, color vertex u_{3i}, $i \in \{1, 2, \ldots, \Delta(H) + 1\}$, with color $\Delta(H) + 1 + i$ and all its neighbors in fiber $H^{u_{3i}}$ with the remaining colors of C_2, and finally color vertex u_{3i+1}, $i \in \{1, 2, \ldots, \Delta(H)+1\}$, with color $2\Delta(H)+2+i$ and all its neighbors in fiber $H^{u_{3i+1}}$ with the remaining colors of C_3. To complete the b-coloring color vertex u_1 and all its neighbors in fiber H^{u_1} with colors C_3 and vertex $u_{3(\Delta(H)+1)+2}$ and its neighbors in fiber $H^{u_{3(\Delta(H)+1)+2}}$ with colors C_1. Note that H-fibers that receive colors from the same C_i color class are at distance 3 and therefore not adjacent. The remaining vertices, if there are any, can be colored with the greedy algorithm. Hence we have a b-coloring of $P_n \boxtimes H$ with $3(\Delta(H) + 1)$ colors and the result follows from inequality (1).

The proof for the cycle is the same. There is only one additional condition for the construction of the b-coloring when $n = 3(\Delta(H) + 1) + 2$. Namely, colors in H^{u_1}-fiber must be different as the colors of H^{u_n}-fiber. □

Note that when there is more than one vertex of maximum degree in H, the bound for n can be even smaller.

Further we consider the case when $H = P_k$ and $H = C_k$. Note that $\Delta(P_k)$ is different whether $k = 1$, $k = 2$, or $k \geq 3$. Applying Theorem 2.3 we get:

Corollary 2.4 *Let P_n and P_k be paths on n and k vertices and C_m and C_ℓ be cycles on n and k vertices respectively. Then*

(i) $\varphi(P_n \boxtimes P_1) = 3$, $n \geq 5$,
(ii) $\varphi(P_n \boxtimes P_2) = 6$, $n \geq 8$,
(iii) $\varphi(P_n \boxtimes P_k) = 9$, $n \geq 11, k \geq 3$,
(iv) $\varphi(C_m \boxtimes C_\ell) = 9$, $m \geq 9, \ell \geq 3$.

For the smaller cases one can check the following. If $\varphi(P_n \boxtimes P_k)$ contains subgraph $P_5 \boxtimes P_3$ then its b-chromatic number is 9, if it only contains subgraph $P_4 \boxtimes P_2$ then its b-chromatic number is 6 and if it only contains subgraph $P_2 \boxtimes P_2$ then its b-chromatic number is 4.

Next we consider cases when $H = K_{1,k}$. Next result follows directly from the Theorem 2.3:

Corollary 2.5 *Let $1 \leq k \leq \lfloor \frac{n-5}{3} \rfloor$. Then*

$$\varphi(P_n \boxtimes K_{1,k}) = 3(k+1) = \varphi(C_n \boxtimes K_{1,k}).$$

Proof. By Theorem 2.3 we have $\varphi(G \boxtimes K_{1,k}) = 3(k+1)$, if G is a path or cycle on n vertices and n is large enough. We take the same lower bound for n as in the proof of theorem 2.3, that is $n \geq 3(k+1)+2$, which leads to $k \leq \lfloor \frac{n-5}{3} \rfloor$. □

We note that there are many cases in Corollary 2.5 that have not been checked. What happens when n is smaller then k? Partially we answer this question in the next two propositions:

Proposition 2.6 *If P_n is the path on n vertices and $K_{1,k}$ a star on $k+1$ vertices. Then*

$$\varphi(P_1 \boxtimes K_{1,k}) = 2,\ k \geq 1$$
$$\varphi(P_2 \boxtimes K_{1,k}) = 4,\ k \geq 1$$
$$\varphi(P_3 \boxtimes K_{1,k}) = \begin{cases} 4 & \text{for} \quad k = 1 \\ 5 & \text{for} \quad k = 2 \\ 6 & \text{for} \quad k \geq 3 \end{cases}$$
$$\varphi(P_4 \boxtimes K_{1,k}) = \begin{cases} 4 & \text{for} \quad k = 1 \\ 6 & \text{for} \quad k \geq 2 \end{cases}$$
$$\varphi(P_5 \boxtimes K_{1,k}) = 6,\ k \geq 1$$
$$\varphi(P_n \boxtimes K_{1,k}) = n,\ k \geq n-2 \geq 4$$

Proof. Note that $P_1 \boxtimes K_{1,k}$ is isomorphic to $K_{1,k}$ and the first assertion is clear. We have

$$4 = \varphi(P_2)\varphi(K_{1,k}) \leq \varphi(P_2 \boxtimes K_{1,k}) \leq m(P_2 \boxtimes K_{1,k}) = 4$$

for $k \geq 1$ and the second assertion is also done. Sporadic examples $P_3 \boxtimes K_{1,1}$, $P_3 \boxtimes K_{1,2}$ and $P_4 \boxtimes K_{1,1}$ are easy and left to the reader.

For all other cases note that $m(P_n \boxtimes K_{1,k})$ is the achieved upper bound. Thus we only need to construct appropriate b-colorings with $m(P_n \boxtimes K_{1,k})$ colors. In the following schemes we present the b-colorings of $P_3 \boxtimes K_{1,k}$ for $k \geq 3$, $P_4 \boxtimes K_{1,k}$ for $k \geq 2$, $P_5 \boxtimes K_{1,k}$ for $k \geq 1$, and $P_n \boxtimes K_{1,k}$ for $k \geq n-2 \geq 4$, respectively:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 5 & 4 & 6 \\ 6 & 5 & 4 \\ 4 & 6 & 5 \\ \vdots & \vdots & \vdots \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 1 \\ 3 & 6 & 5 & 2 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \begin{bmatrix} 5 & 1 & 3 & 5 & 1 \\ 6 & 2 & 4 & 6 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 3 & 4 & 5 & \dots & 1 & 3 \\ 4 & 5 & 6 & \dots & 2 & 4 \\ 5 & 6 & 7 & \dots & 3 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-1 & n & 1 & \dots & n-3 & 1 \\ n & 4 & 5 & \dots & 1 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Note that each line presents a P_n-fiber, while columns present $K_{1,k}$-fibers. Furthermore, first line is due to the center vertex of the star. □

Next we consider strong products of cycles and stars. One would suspect that the result would be similar to the one of the strong product of paths and stars, but that is not always the case.

Proposition 2.7 *If C_n is the cycle on n vertices and $K_{1,k}$ a star on $k+1$ vertices. Then*

$$\varphi(C_3 \boxtimes K_{1,k}) = 6,\ k \geq 1,$$

$$\varphi(C_4 \boxtimes K_{1,k}) = \begin{cases} 4 & \textit{for} \quad k = 1 \\ 5 & \textit{for} \quad k = 2,3 \\ 6 & \textit{for} \quad k \geq 4 \end{cases},$$

$$\varphi(C_5 \boxtimes K_{1,k}) = 6,\ k \geq 1,$$

$$\varphi(C_n \boxtimes K_{1,k}) = n,\ k \geq n-3 \geq 3.$$

Proof. The proof is analogue to the previous proof in most cases. Thus we only give a slightly different scheme for the b-coloring for $C_4 \boxtimes K_{1,k}$, $k \geq 4$, and for the last case

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 5 & 4 & 6 & 4 \\ 6 & 5 & 4 & 5 \\ 4 & 6 & 5 & 6 \\ 3 & 4 & 1 & 5 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 3 & 4 & 5 & \dots & 1 & 2 \\ 4 & 5 & 6 & \dots & 2 & 4 \\ 5 & 6 & 7 & \dots & 3 & 5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n-1 & n & 1 & \dots & n-3 & n-2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and make a note for $C_4 \boxtimes K_{1,1}$, $C_4 \boxtimes K_{1,2}$ and $C_4 \boxtimes K_{1,3}$ since the upper bound is not achieved. Suppose that you can find a 5-b-coloring of $C_4 \boxtimes K_{1,1}$. By symmetry every vertex can be a color 1 dominating vertex. Then we have one of the following schemes (by the permutations of the color classes):

$$\begin{bmatrix} 1 & 3 & \_ & 5 \\ 2 & 4 & \_ & \_ \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 3 & \_ & \_ \\ 2 & 4 & \_ & 5 \end{bmatrix}.$$

However there is no color 5 dominating vertex in both cases, which is not possible for a b-coloring. It is easy to find a 4-b-coloring of $C_4 \boxtimes K_{1,1}$.

Now let us consider graphs $C_4 \boxtimes K_{1,k}$, $k = 2, 3$. Suppose there exists a 6-b-coloring ob both graphs. If the first line of the b-coloring scheme (the line due to the center of the star) contains two or four different colors then it is easy to see that every other line of the matrix can not have a color dominating vertex. Now suppose that the first line of the scheme contains three different colors. Then every other line can contain at most one color dominating vertex. Since we need at least 6 dominating vertices, k must be at least 3. Moreover, the three vertices in the first layer having different colors must also be color dominating and hence this can only happen if k is at least 4. A 5-b-coloring of graphs $C_4 \boxtimes K_{1,k}$, $k = 2, 3$, is shown in the following schemes:

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 1 & 5 \\ 3 & 4 & 5 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 5 & 1 & 5 \\ 3 & 4 & 5 & 4 \\ 4 & 5 & 1 & 5 \end{bmatrix}.$$

□

Propositions 2.6 and 2.7 show that the b-chromatic number of a strong product is not always bounded by the b-chromatic number of its factors. Take for example P_n and $K_{1,n-2}$, $n \geq 6$. Then $\varphi(P_n) = 3$ and $\varphi(K_{1,n-2}) = 2$, but $\varphi(P_n \boxtimes K_{1,n-2}) = n$.

One might try to find an upper bound for $\varphi(G \boxtimes H)$ with respect to $m(G)$ and $m(H)$, however this is again not possible. Take the same graphs then in the above argument, that is P_n and $K_{1,n-2}$, $n \geq 6$. Then $m(P_n) = 3$ and $m(K_{1,n-2}) = 2$, but $\varphi(P_n \boxtimes K_{1,n-2}) = n$. Since $m(G \boxtimes H)$ is an upper bound of $\varphi(G \boxtimes H)$, which means that $m(G \boxtimes H) \geq n$, and hence, can not be a function of $m(G)$ and $m(H)$.

We conclude this section with a conjecture of an upper bound:

Conjecture 2.8 *Let G and H be two arbitrary graphs. Then*

$$\varphi(G \boxtimes H) \leq \max\{\varphi(G)(\Delta(H) + 1), (\Delta(G) + 1)\varphi(H)\}$$

3 Lexicographic product

In this section we concentrate on the lexicographic product. We show a strict lower bound and that there is no upper bound that depends only on the b-chromatic numbers of the factors. On the way we derive many exact results and give a conjecture for the upper bound.

The lower bound for the lexicographic product is the same as for the strong product, even more the proof is the same once we exchange the strong product with lexicographic whenever it occurred in the proof of Proposition 2.1. Hence:

Proposition 3.1 *Let G and H be arbitrary graphs. Then*

$$\varphi(G[H]) \geq \varphi(G)\varphi(H)\,.$$

The above lower bound is strict as we will see on some small examples ($P_2[P_m]$, $m \in \{3, 4\}$). However if the graphs are "bigger" this lower bound does not perform so well. Again by the trivial upper bound for the b-chromatic number we have:

$$\varphi(G[H]) \leq \Delta(G)|V(H)| + \Delta(H) + 1\,. \tag{2}$$

Next we concentrate on some exact results. Let first both factors be paths $P_n[P_m]$ where $P_n = u_1u_2 \ldots u_n$ and $P_m = v_1v_2 \ldots v_m$. By the above upper bound we have $\varphi(P_n[P_m]) \leq 2m + 3$. Let first $m \geq 5$. Then we can color the fiber $P_m^{u_{2k}}$, $k \in \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$ with three colors $3(k-1)+i$, $i \in \{1, 2, 3\}$, such that $P_m^{u_{2k}}$ contains the following sequence of these three colors $3k, 3k-1, 3k-2, 3k, 3k-1 \ldots$. The neighboring fibers $P_m^{u_{2k-1}}$ and $P_m^{u_{2k+1}}$ then get $2m$ remaining colors (each vertex a different color). We have an additional condition that colors $3k+i$, $i \in \{1, 2, 3\}$, are in the $P_m^{u_{2k-1}}$-fiber, so that they can be in the next $P_m^{u_{2k+2}}$ fiber. Clearly this coloring is a proper coloring and colors $3(k-1)+i$, $i \in \{1, 2, 3\}$, have they dominating vertices in $P_m^{u_{2k}}$-fibers if and only if n is big enough. Since every second P_m-fiber contains three color-dominating vertices and we need the last fiber we have $n \geq 2\left(\left\lceil \frac{2m+3}{3} \right\rceil\right) + 1 = 2\left\lceil \frac{2m}{3} \right\rceil + 3$. Thus we have proved:

Theorem 3.2 *Let $m \geq 5$ and $n \geq 2\left\lceil \frac{2m}{3} \right\rceil + 3$. Then $\varphi(P_n[P_m]) = 2m+3$.*

The following scheme represent a b-coloring with 15 colors of $P_{11}[P_6]$:

$$\begin{bmatrix} 1 & 7 & 10 & 1 & 4 & 10 & 1 & 4 & 7 & 13 & 1 \\ 2 & 8 & 11 & 2 & 5 & 11 & 2 & 5 & 8 & 14 & 2 \\ 3 & 9 & 12 & 3 & 6 & 12 & 3 & 6 & 9 & 15 & 3 \\ 4 & 7 & 13 & 1 & 7 & 10 & 13 & 4 & 10 & 13 & 4 \\ 5 & 8 & 14 & 2 & 8 & 11 & 14 & 5 & 11 & 14 & 5 \\ 6 & 9 & 15 & 3 & 9 & 12 & 15 & 6 & 12 & 15 & 6 \end{bmatrix}.$$

For $m = 4$ the construction is similar, with the difference that there are at most two color-dominating vertices in a $P_m^{u_{2k}}$-fiber. Thus:

Proposition 3.3 *Let $n \geq 13$. Then $\varphi(P_n[P_4]) = 11$.*

Colorings for the following results are easy to obtain as well as the upper bounds and are left to the reader:

$$\varphi(P_n[P_2]) = \begin{cases} 2 & \text{for} \quad n = 1 \\ 4 & \text{for} \quad n = 2,3,4 \\ 6 & \text{for} \quad n \geq 5 \end{cases} \quad \text{and } \varphi(P_n[P_3]) = \begin{cases} 4 & \text{for} \quad n = 2,3,4 \\ 5 & \text{for} \quad n = 5 \\ 6 & \text{for} \quad n = 6,7,8 \\ n-2 & \text{for} \quad n = 9,10 \\ 9 & \text{for} \quad n \geq 11 \end{cases}.$$

Similar idea can be used for a lexicographic product of a path and an arbitrary graph H.

Theorem 3.4 *Let H be an arbitrary graph. Then $\varphi(P_n[H]) = 2|V(H)| + \Delta(H) + 1 = \varphi(C_n[H])$, for each $n \geq 2\,(2|V(H)| + \Delta(H) + 1) + 1$.*

Proof. Let $n \geq 2\,(2|V(H)| + \Delta(H) + 1) + 1$. By inequality (2) we have $\varphi(P_n[H]) \leq 2|V(H)| + \Delta(H) + 1$. Thus we only have to describe a b-coloring with $2|V(H)| + \Delta(H) + 1$ colors for graph $P_n[H]$. Let $k = 2\,(2|V(H)| + \Delta(H) + 1) + 1$ and let $n \geq k$. Furthermore let $v \in V(H)$ be a vertex of maximum degree in H and let $P_n = u_1 u_2 \dots u_n$. We define a vertex coloring c : let $c((u_{2i}, v)) = i$, for $i \in \{1, 2, \dots, 2|V(H)| + \Delta(H) + 1\}$, and every vertex from the neighborhood of (u_{2i}, v) in $P_n[H]$ receives a different color with additional condition that color $i+1$ is not in $H^{u_{2i+1}}$-fiber. All the remaining vertices (if there are any) can be colored by the greedy algorithm. Clearly this is a proper coloring since there are $|V(H)|$ different colors in every $H^{u_{2i+1}}$-fiber and $\Delta(H) + 1$ different colors in every $H^{u_{2i}}$-fiber. Moreover this is a b-coloring since (u_{2i}, v) is a color i dominating vertex for every color i.

As in the proof of Theorem 2.3 is the proof for cycles the same with the additional condition when $n = 2\,(2|V(H)| + \Delta(H) + 1) + 1$. Namely, colors in H^{u_1}-fiber must be different as the colors of H^{u_n}-fiber. $\square$

Note that if there is more than one vertex of maximum degree in H, the bound for n in the above proof can be smaller.

Next we suppose that the first factor is a complete bipartite graph $K_{n,m}$ with the partition A' and B' with $|A'| = n$ and $|B'| = m$. Then for any graph H the vertex set of $K_{n,m}[H]$ can be split into two sets $A = \{(u,v)|u \in A', v \in V(H)\}$ and $B = \{(u,v)|u \in B', v \in V(H)\}$. Let c be a proper coloring of $K_{n,m}[H]$. Then each color i can appear either completely in A or completely in B. Let (u,v) be a color i-dominating vertex from A. Then it has as neighbors all vertices from B and as such also all colors that appear in B. On the other hand (u,v) has only $\deg_H(v)$ neighbors in A and can dominate at most $\deg_H(v)$ colors in A. Thus there can be at most $\Delta(H)+1$ different colors in A in any b-coloring of $K_{n,m}[H]$ and additional $\Delta(H)+1$ different colors in B. This yields the upper bound $\varphi(K_{n,m}[H]) \leq 2(\Delta(H)+1)$. In addition we can have, if m and n are big enough ($\geq \Delta(H)+1$), exactly one dominating vertex in each H-fiber—the vertex of maximum degree in H. Hence, we have proved the following:

Theorem 3.5 *Let H be a graph and $m, n \geq \Delta(H)+1$. Then $\varphi(K_{n,m}[H]) = 2(\Delta(H)+1)$.*

In the special case when also H is a complete bipartite graph $K_{\ell,k}$ (recall that $\ell \geq k$) we obtain the next result.

Corollary 3.6 *For any $\ell \geq k > 0$ and $n \geq m > 0$ we have*

$$\varphi(K_{n,m}[K_{\ell,k}]) = \min\{m, \ell+1\} + \min\{n, \ell+1\}\,.$$

Proof. Note that $\Delta(K_{\ell,k}) = \ell$ and if $n \geq m > \ell$ there is nothing to prove by Theorem 3.5. If either $n \geq \ell \geq m$ or $\ell \geq n \geq m$ occurred, note that in notation used before previous theorem in A can be at most $\ell+1$ or n, respectively, different colors and in B can be at most m different colors. Hence the result. $\square$

Hence for a positive integer n we have

$$\varphi(K_{n,n}[K_{n,n}]) = 2n. \tag{3}$$

As in the previous section there is no upper bound for $\varphi(G[H])$ with respect to the factors $\varphi(G)$ and $\varphi(H)$. By equation (3) the proof is by nothing since $\varphi(K_{n,n}) = 2$ and n can be arbitrary.

Also there is no upper bound for $\varphi(G[H])$ and $m(G[H])$ with respect to the factors $m(G)$ and $m(H)$. Let n be such that $G = P_n$ and $H = K_{1,k}$ fulfill the

Theorem 3.4. Then $\varphi(P_n[K_{1,k}]) = 3k+3$ by the same theorem and since $m(P_n) = 3$ and $m(K_{1,k}) = 2$, $\varphi(G[H])$ is not a function of $m(G)$ and $m(H)$. Also, since m is an upper bound of φ, $m(G[H])$ is not a function of $m(G)$ and $m(H)$.

Is there any other general upper bound? We strongly suspect that the following holds:

Conjecture 3.7 *For any graphs G and H we have $\varphi(G[H]) \le (\varphi(G) - 1)\,|V(H)| + \Delta(H) + 1$.*

4 Direct product

In the last section we give a lower bound for the direct product. It is easy to see the following:

Proposition 4.1 *Let G and H be two arbitrary graphs. Then*

$$\varphi(G \times H) \ge \max\{\varphi(G), \varphi(H)\}\,.$$

Proof. Let $\varphi(G) \ge \varphi(H)$. Then color all G-fibers in $G \times H$ the same as one would color graph G with a $\varphi(G)$-b-coloring. The coloring is obviously proper and every color dominating vertex in G is still a color dominating vertex in $\varphi(G \times H)$, even though it has neighbors in other G-fibers, not the one he is lying in. $\square$

The above lower bound is analogue to the upper bound for the chromatic number of a direct product

$$\chi(G \times H) \le \min\{\chi(G), \chi(H)\},$$

which gives base stone for the Hedetniemi's conjecture $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. The same does not hold for the b-chromatic number of the direct product. Take for example graph $P_5 \times P_5$. This graph has a subgraph $P_4 \,\square\, P_2$, therefore $\varphi(P_5 \times P_5) \ge \varphi(P_4 \,\square\, P_2) = 4$, and $\varphi(P_5) = 3$. Hence $\varphi(P_5 \times P_5) > \varphi(P_5)$.

On the other hand this lower bound is strict as can be seen from a family $K_2 \times P_n$ for every positive integer n.

Again we have a trivial upper bound for the direct product:

$$\varphi(G \times H) \le \Delta(G)\Delta(H) + 1. \tag{4}$$

Theorems 2.3 and 3.4 have an analogue also for the direct product:

Theorem 4.2 *Let H be a graph with at least one edge. Then $\varphi(P_n \times H) = 2\Delta(H) + 1$, for every $n \ge 2\,(2\Delta(H) + 1) + 1$.*

Proof. By inequality (4) we have $\varphi(P_n \times H) \leq 2\Delta(H) + 1$. Thus we only have to describe a b-coloring with $2\Delta(H) + 1$ colors for graph $P_n \times H$. Let $n \geq 2\,(2\Delta(H) + 1) + 1$. Furthermore let $v \in V(H)$ be a vertex of a maximum degree in H and let $P_n = u_1 u_2 \ldots u_n$. We define a vertex coloring c : let $c((u_{2i}, v)) = i$, for $i \in \{1, 2, \ldots, 2\Delta(H) + 1\}$, and every vertex from the neighborhood of (u_{2i}, v) in $P_n \times H$ receives a different color with additional condition that color $i + 1$ is not in $H^{u_{2i+1}}$-fiber. All the remaining vertices (if there are any) can be colored by the greedy algorithm. Clearly this is a proper coloring. Moreover this is a b-coloring since (u_{2i}, v) is a color i dominating vertex for every color i. $\square$

As in the case of the strong and the lexicographic product, the Theorem 4.2 shows that the b-chromatic number of a direct product is not always bounded by the b-chromatic number of its factors. Take P_{4k+3} and $K_{1,k}$, $k \geq 1$. Then $\varphi(P_{4k+3}) = 3$ and $\varphi(K_{1,k}) = 2$, but $\varphi(P_{4k+3} \times K_{1,k}) = 2k + 1$.

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