The k-path vertex cover of rooted product graphs

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Abstract

A subset S of vertices of a graph G is called a k-path vertex cover if every path of order k in G contains at least one vertex from S. Denote by $\psi_k(G)$ the minimum cardinality of a k-path vertex cover in G. In this article a lower and an upper bound for ψ_k of the rooted product graphs are presented. Two characterizations are given when those bounds are attained. Moreover ψ_2 and ψ_3 are exactly determined. As a consequence the independence and the dissociation number of the rooted product are given.

Keywords: k-path vertex cover, vertex cover, independence number, dissociation number, rooted product
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1. Introduction and preliminaries

Let G be a simple, undirected graph. For a positive integer $k \geq 2$ the subset $S \subseteq V(G)$ is a *k-path vertex cover of* G, if every path of order k in graph G contains a vertex from S. The set S is also called the set of covered vertices in a k-path vertex cover of G and we call $T = V(G) - S$ the set of uncovered vertices. The cardinality of a minimum k-path vertex cover is denoted by $\psi_k(G)$.

The motivation for this invariant was introduced in [4] and arises from communications in wireless sensor networks, where the data integrity is ensured by using the Novotný's k-generalized Canvas scheme [15]. There are many other motivations, for instance in traffic control as presented in [20].

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The problem of computing $\psi_k(G)$ is in general NP-hard for each $k \geq 2$, but it was also shown that it is polynomial for trees. In [19, 20, 21] some approximation algorithms for $\psi_3(G)$ were derived and in [13] an exact algorithm for computing $\psi_3(G)$ in running time $O(1.5171^n)$ for a graph of order n was presented.

The k-path vertex cover is a generalization of *the vertex cover*. It is easy to see that $\psi_2(G)$ equals the size of a minimum vertex cover. Moreover,

$$\psi_2(G) = |V(G)| - \alpha(G)\,,$$

where $\alpha(G)$ denotes the maximum stable set and is called *the independence number* of G. This gives an interesting connection to the well studied independence number [10, 11, 18, 22].

The value $\psi_3(G)$ is in close relation to the concept of the *dissociation number* of a graph [23]. A subset of vertices in a graph G is called a *dissociation set* if it induces a subgraph with maximum degree at most 1. The number of vertices in a maximum cardinality dissociation set in G is called the dissociation number of G and is denoted by $\mathrm{diss}(G)$. The relation between $\psi_3(G)$ and $\mathrm{diss}(G)$ is

$$\psi_3(G) = |V(G)| - \mathrm{diss}(G)\,.$$

Determining the dissociation number of a graph is NP-hard in the class of bipartite graphs [23]. The dissociation number problem was also studied in several other articles [1, 2, 5, 9]. This results were also united in a survey, see [16].

Recently, in [3] some results on d-regular graphs were presented. For instance for an arbitrary integer $k \geq 2$ and a d-regular graph G, $d \geq k-1$, it follows that

$$\psi_k(G) \geq \frac{d-k+2}{2d-k+2}|V(G)|\,.$$

The concept of the k-path vertex cover was also studied in different graph products. In [3] the exact value for ψ_3 was determined for the Cartesian product of paths. Also, some bounds for the same products were determined for ψ_k. These bounds were later improved in [12] and extended to the strong product of paths. In the same article [12] some results for the lexicographic product were presented, which were the first results in graph products for arbitrary graphs. A good lower and upper bound for the lexicographic product of arbitrary graphs was given and the exact value for ψ_2 and ψ_3 was determined. As a consequence, the independence and the dissociation number of the lexicographic product were derived. Those results imply a well-known result of Geller and Stahl [7] who determined the independence number of the lexicographic product.

We continue our research in the rooted product which is closely related to the Cartesian product. The rooted product of graphs was studied on many occasions, see for instance [6, 8, 14, 17, 24]. Since no results for the k-path vertex cover of the Cartesian product of arbitrary graphs were presented it would be interesting to see if some general results can be derived in the rooted product of graphs.

Let $V(G) = \{g_1, g_2, \ldots, g_m\}$ and $V(H) = \{h_1, h_2, \ldots, h_n\}$. We choose a vertex from $V(H)$ to be the root vertex of H, say h_1. The *rooted product* $G \circ H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ has the vertex set

$$V(G \circ H) = \{(g_i, h_j) \,|\, i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\},$$

and the edge set

$$E(G \circ H) = \{(g_i, h_1)(g_k, h_1) \,|\, g_i g_k \in E(G)\} \cup \bigcup_{i=1}^{m} \{(g_i, h_j)(g_i, h_k) \,|\, h_j h_k \in E(H)\}.$$

If G is also rooted at g_1, one can view the product itself as rooted at (g_1, h_1). The rooted product is a subgraph of the Cartesian product of the same two graphs. An example of the rooted product $C_3 \circ P_4$, where P_4 is rooted at an inner vertex, can be seen in Figure 1.

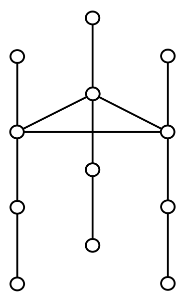

Figure 1: A rooted product $C_3 \circ P_4$

Let G and H be arbitrary graphs and H rooted at h. We refer to the set $V(G) \times \{h\}$ as the G-layer of graph $G \circ H$. Similarly, for any vertex

$u \in V(G)$, the set $\{u\} \times V(H)$ is an H-layer. Note that there is only one G-layer, but there might be many H-layers. Layers can also be regarded as the graphs induced on the sets that define them. Obviously, in the rooted product the G-layer and an H-layer are isomorphic to G and H, respectively.

Since the main motivation for the k-path vertex cover is securing networks with as few sensors as possible one can view the rooted product as a combination of many local networks (copies of graph H) having a server (the root vertex of graph H). These servers are connected through a global network (the graph G). Hence, we get another motivation why it is interesting to study the k-path vertex cover of rooted product graphs.

2. Main results

We start this section with the following results of a lower and an upper bound of $\psi_k(G \circ H)$. Note that in the figures the vertices which belong to a k-path vertex cover S are colored black.

Proposition 2.1. *Let G and H be arbitrary connected graphs and H rooted at any vertex $h \in V(H)$. Then*

$$|V(G)|\psi_k(H) \leq \psi_k(G \circ H) \leq |V(G)|\psi_k(H) + \psi_2(G)\,.$$

Proof. We need at least $\psi_k(H)$ covered vertices in every H-layer of graph $G \circ H$. The number of such layers is $|V(G)|$ therefore the lower bound of $\psi_k(G \circ H)$ is $|V(G)|\psi_k(H)$.

For the upper bound we construct a k-path vertex cover of graph $G \circ H$. Let S_1 be a minimum k-path vertex cover of graph H and S_2 a minimum 2-path vertex cover (i.e. vertex cover) of graph G. If every H-layer is covered in the same way as S_1 covers graph H and the G-layer is covered in the same way as S_2 covers graph G then the size of the minimum k-path vertex cover of graph $G \circ H$ is at most

$$\left(\sum_{i=1}^{|V(G)|} |S_1|\right) + |S_2|\,.$$

Suppose this is not true. Then we have a path of order k in $G \circ H$ which is not covered. If such a path lies in more than one H-layer then there exist two adjacent vertices on this path that lie in the G-layer of graph $G \circ H$. Hence, we have two adjacent vertices in the G-layer that are not covered. This is a contradiction since S_2 is a vertex cover of graph G. Therefore, this path lies completely in one of the H-layers which is also a contradiction since S_1 is a k-path vertex cover of graph H. Hence

$$\begin{aligned}\psi_k(G \circ H) &\leq \left(\sum_{i=1}^{|V(G)|} |S_1|\right) + |S_2| \\ &= |V(G)||S_1| + |S_2| = |V(G)|\psi_k(H) + \psi_2(G)\,.\end{aligned}$$

□

Having proved Proposition 2.1 it would be interesting to know when both bounds are achieved. For the sake of this we introduce the following definition.

Definition 2.2. *Let H be an arbitrary graph and $h \in V(H)$. If there exists a minimum k-path vertex cover S of graph H, i.e. $|S| = \psi_k(H)$, such that $h \in S$, then we call vertex h a kPVC-perfect vertex.*

As an example (Figure 2), we take the path $P_{k+1} = v_1v_2v_3\ldots v_kv_{k+1}$ for which $\psi_k(P_{k+1}) = 1$. It is easy to see that all vertices except v_1 and v_{k+1} are kPVC-perfect vertices.

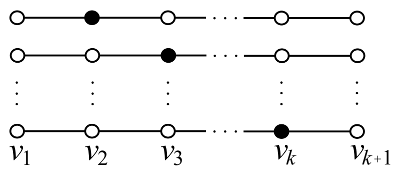

Figure 2: The path P_{k+1} with all possibilities for the kPVC-perfect vertex

With the help of Definition 2.2 we can give a sufficient condition when the lower bound in Proposition 2.1 is achieved.

Theorem 2.3. *Let G and H be arbitrary connected graphs, graph H rooted at $h \in V(H)$, and $\psi_k(G) \neq 0$. Then*

$$\psi_k(G \circ H) = |V(G)|\psi_k(H)$$

if and only if h is a kPVC-perfect vertex.

Proof. Suppose that $h \in V(H)$ is a kPVC-perfect vertex. Then there exists a minimum k-path vertex cover S of graph H such that $h \in S$. We construct a k-path vertex cover of graph $G \circ H$ in such way that we cover every H-layer in the same way as S covers graph H. In this sense, vertices (g_i, h), $i \in \{1, \dots, |V(G)|\}$, are covered since the vertex h is a kPVC-perfect vertex. Hence, the G-layer is completely covered and there is no uncovered path of order k in the $G \circ H$ having some of its vertices in the G-layer. Also, since S is a k-path vertex cover of H there is no path of order k in any H-layer. Hence,

$$\psi_k(G \circ H) \leq |V(G)||S| = |V(G)|\psi_k(H)\,.$$

According to Proposition 2.1 this upper bound is also the lower bound of $\psi_k(G \circ H)$ and therefore

$$\psi_k(G \circ H) = |V(G)|\psi_k(H)\,.$$

For the converse, suppose that $\psi_k(G \circ H) = |V(G)|\psi_k(H)$. Let S be a minimum k-path vertex cover of graph $G \circ H$ and $S_i = \{(g_i, h_j) \in S \,|\, j \in \{1, \dots, |V(H)|\}\}$, $i \in \{1, \dots, |V(G)|\}$, the set of vertices of S that lie in the H_i-layer. Hence,

$$|S| = \sum_{i=1}^{|V(G)|} |S_i|\,.$$

Clearly, $|S_i| \geq \psi_k(H)$. If $|S_i| > \psi_k(H)$, $i \in \{1, \dots, |V(G)|\}$, then

$$|S| = \sum_{i=1}^{|V(G)|} |S_i| > \sum_{i=1}^{|V(G)|} \psi_k(H) = |V(G)|\psi_k(H)$$

which is a contradiction. Therefore, $|S_i| = \psi_k(H)$, for any $i \in \{1, \dots, |V(G)|\}$. Suppose that h is not a kPVC-perfect vertex. Then h does not lie in any minimum k-path vertex cover of graph H. Each set S_i, $i \in \{1, \dots, |V(G)|\}$, is a minimum k-path vertex cover of the H_i-layer, $i \in \{1, \dots, |V(G)|\}$. Since h is not a kPVC-perfect vertex, $(g_i, h) \notin S_i$, for all $i \in \{1, \dots, |V(G)|\}$. Moreover, $(g_i, h) \notin S$, for all $i \in \{1, \dots, |V(G)|\}$. This means that the G-layer of graph $G \circ H$, which is isomorphic to graph G, is completely uncovered. This is a contradiction to the assumption that $\psi_k(G) \neq 0$. Hence, h is a kPVC-perfect vertex. □

Remark 2.4. *The assumption $\psi_k(G) \neq 0$ in Theorem 2.3 is only needed to prove one implication. Hence, when h is a kPVC-perfect vertex it always holds that $\psi_k(G \circ H) = |V(G)|\psi_k(H)$, even if $\psi_k(G) = 0$.*

With the help of Theorem 2.3 and Remark 2.4 we can prove the following results.

Proposition 2.5. *Let G and H be arbitrary connected graphs and H rooted at $h \in V(H)$. If h is not a kPVC-perfect vertex then*

$$\psi_k(G \circ H) \geq |V(G)|\psi_k(H) + \psi_k(G)\,.$$

Proof. If $\psi_k(G) = 0$, the result is the same as in the Proposition 2.1. Suppose that $\psi_k(G) \neq 0$. By Theorem 2.3 we know that $\psi_k(G \circ H) \geq |V(G)|\psi_k(H) + 1$. We can show more, namely that $\psi_k(G \circ H) \geq |V(G)|\psi_k(H) + \psi_k(G)$. If H is the vertex graph, then this bound is trivial, since $\psi_k(G \circ H) = \psi_k(G)$ and $\psi_k(H) = 0$. Let H be different than the vertex graph. Without loss of generality, suppose that the root vertex h is the vertex h_1. Let S be a minimum k-path vertex cover of graph $G \circ H$, $S_i = \{(g_i, h_j) \in S \,|\, j \in \{2, \ldots, |V(H)|\}\}$, $i \in \{1, \ldots, |V(G)|\}$, and $S' = \{(g_i, h_1) \in S \,|\, i \in \{1, \ldots, |V(G)|\}\}$. It is obvious that

$$|S| = \left(\sum_{i=1}^{|V(G)|} |S_i| \right) + |S'|\,.$$

Since h_1 is not a kPVC-perfect vertex, every minimum k-path vertex cover of H does not contain vertex h_1. If $|S_i| = \psi_k(H) - 1$, for some i, then $S_i \cup \{(g_i, h_1)\}$ is a minimum k-path vertex cover of graph induced by the H_i-layer, and hence h_1 is a kPVC-perfect vertex, which is not possible. Hence, $|S_i| \geq \psi_k(H)$, for all $i \in \{1, \ldots, |V(G)|\}$. Also, $|S'| \geq \psi_k(G)$. Therefore,

$$\begin{aligned} |S| &= \left(\sum_{i=1}^{|V(G)|} |S_i| \right) + |S'| \\ &\geq \left(\sum_{i=1}^{|V(G)|} \psi_k(H) \right) + \psi_k(G) \\ &= |V(G)|\psi_k(H) + \psi_k(G)\,, \end{aligned}$$

and the proof is complete. □

Corollary 2.6. *Let G and H be arbitrary connected graphs and H rooted at $h \in V(H)$. Then*

$$\psi_2(G \circ H) = \begin{cases} |V(G)|\psi_2(H) & ; \; h \text{ is a } kPVC\text{-perfect vertex} \\ |V(G)|\psi_2(H) + \psi_2(G) & ; \; h \text{ is not a } kPVC\text{-perfect vertex} \end{cases}.$$

Proof. If G is the vertex graph, then $\psi_2(G) = 0$. It follows that

$$\psi_2(G \circ H) = \psi_2(H) = |V(G)|\psi_2(H) = |V(G)|\psi_2(H) + \psi_2(G)$$

and both results coincide no matter whether h is kPVC-perfect or not.

Suppose now that G is different from the vertex graph. Since G is connected, $\psi_2(G) \neq 0$. By Theorem 2.3 $\psi_2(G \circ H) = |V(G)|\psi_2(H)$ if and only if h is a kPVC-perfect vertex. If h is not a kPVC-perfect vertex then by Proposition 2.5 it follows that

$$|V(G)|\psi_2(H) + \psi_2(G) \leq \psi_2(G \circ H) \leq |V(G)|\psi_2(H) + \psi_2(G)\,,$$

and hence, $\psi_2(G \circ H) = |V(G)|\psi_2(H) + \psi_2(G)$. □

Corollary 2.7. *Let G and H be arbitrary connected graphs and H rooted at $h \in V(H)$. Then*

$$\alpha(G \circ H) = \begin{cases} |V(G)|\alpha(H) & ; \ h \text{ is a } kPVC\text{-perfect vertex} \\ |V(G)|(\alpha(H) - 1) + \alpha(G) & ; \ h \text{ is not a } kPVC\text{-perfect vertex} \end{cases} .$$

Proof. By Corollary 2.6 the result follows immediately. First, suppose that h is a kPVC-perfect vertex. Then

$$\begin{aligned} \alpha(G \circ H) &= |V(G \circ H)| - \psi_2(G \circ H) \\ &= |V(G)||V(H)| - |V(G)|\psi_2(H) \\ &= |V(G)|(|V(H)| - \psi_2(H)) \\ &= |V(G)|\alpha(H)\,. \end{aligned}$$

If h is not a kPVC-perfect vertex, then

$$\begin{aligned} \alpha(G \circ H) &= |V(G \circ H)| - \psi_2(G \circ H) \\ &= |V(G)||V(H)| - |V(G)|\psi_2(H) - \psi_2(G) \\ &= |V(G)||V(H)| - |V(G)|\psi_2(H) - |V(G)| + |V(G)| - \psi_2(G) \\ &= |V(G)|(|V(H)| - \psi_2(H) - 1) + |V(G)| - \psi_2(G) \\ &= |V(G)|(\alpha(H) - 1) + \alpha(G)\,. \end{aligned}$$

□

The assumption $\psi_k(G) \neq 0$ which is used in Theorem 2.3 and later in the proof of Proposition 2.5 is connected to the fact whether the root vertex is a kPVC-perfect vertex or not. We can derive the following corollary.

Corollary 2.8. *Let G and H be arbitrary connected graphs and H rooted at vertex $h \in V(H)$. If $\psi_k(G \circ H) = |V(G)|\psi_k(H)$ then h is a kPVC-perfect vertex or $\psi_k(G) = 0$.*

Proof. Suppose $\psi_k(G \circ H) = |V(G)|\psi_k(H)$. If $\psi_k(G) = 0$, we are done. We may therefore assume that $\psi_k(G) \neq 0$. By Theorem 2.3 vertex h is a kPVC-perfect vertex. $\square$

Even though Corollary 2.8 is almost the same as Theorem 2.3, it is important to know that the converse in Corollary 2.8 is not true. If h is not a kPVC-perfect vertex and $\psi_k(G) = 0$, then the equality $\psi_k(G \circ H) = |V(G)|\psi_k(H)$ does not necessary hold. Take for example $k \geq 3$, $G = P_{k-1} = u_1u_2 \dots u_{k-1}$ and $H = P_{2k-1} = v_1v_2 \dots v_{2k-1}$ rooted at v_1. It is clear that $\psi_k(G) = 0$. Also, it is easy to see that $\psi_k(H) = 1$ and v_1 is not a kPVC-perfect vertex. There is a unique way how to cover each H-layer with the k-path vertex cover of the size $\psi_k(H) = 1$. However, such a cover is not a k-path vertex cover for the whole graph $G \circ H$ since it is easy to find a path on k vertices which is not covered (see Figure 3). Hence, $\psi_k(G \circ H) > |V(G)|\psi_k(H) = |V(G)|$.

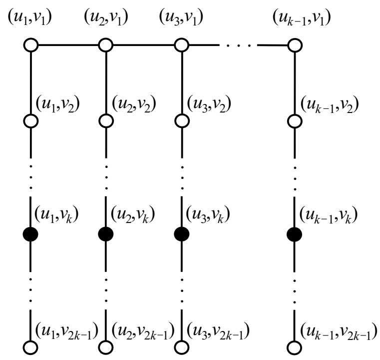

Figure 3: The graph $P_{k-1} \circ P_{2k-1}$ rooted at (u_i, v_1), $i \in \{1, \dots, k-1\}$

We continue our observation by finding conditions for which the value $\psi_k(G \circ H)$ would equal the lower bound in Proposition 2.5 and the upper bound in Proposition 2.1. For both cases we introduce some new definitions.

Let $h \in V(H)$ be a vertex that is not a kPVC-perfect vertex. We may refer to such a vertex as the kPVC-imperfect vertex. We know that $h \notin S$ for any minimum k-path vertex cover S of H. Therefore, $h \in T = V(H) - S$, where T is the set of uncovered vertices. Then vertex h lies in some paths P_i, for $i \in \{1, \dots, k-1\}$, having h as one of its end-vertices, and consisting only of the vertices of the set T. There always exists at least one such path, namely the path P_1 on vertex h. Let $P(H : S : h)$ be the order of the longest path in graph H starting (or ending) in the vertex h such that all vertices of this path are uncovered with respect to S. It is clear that the set depends on graph H, a minimum k-path vertex cover S, and kPVC-imperfect vertex h. To be consistent, we define $P(H : S : h) = 0$ if h is a kPVC-perfect vertex.

Remark 2.9. *Let H be a graph, S a minimum k-path vertex cover of H, and $h \in V(H)$. Then*

$$0 \leq P(H : S : h) \leq k - 1\,.$$

With the help of Remark 2.9 we can define for any graph H the following concept.

Definition 2.10. *Let H be an arbitrary graph and $h \in V(H)$. If*

$$q = \min\{P(H : S : h) \,|\, S \text{ is a minimum } k\text{-path vertex cover of } H\}\,,$$

then we refer to the vertex h as the q-kPVC-imperfect vertex.

For the kPVC-perfect vertex the Definition 2.10 implies that such a vertex is a 0-kPVC-imperfect vertex. To understand the Definition 2.10 we give an example presented in Figure 4. Take again the graph P_{2k-1}. There is a unique way how to cover graph P_{2k-1} with a k-path vertex cover S of size $\psi_k(P_{2k-1}) = 1$. Namely, vertex v_k must be covered. All other vertices are kPVC-imperfect vertices. Hence, $P(H : S : v_1) = k - 1$, and since S is a unique minimum k-path vertex cover, it follows that $q = P(H : S : v_1) = k - 1$. Therefore, v_1 is a $(k - 1)$-kPVC-imperfect vertex. In general, for every v_i, $i \neq k$, we find the longest uncovered path for which v_i is one of its end-vertices. The order of this path equals q and v_i is a q-kPVC-imperfect vertex.

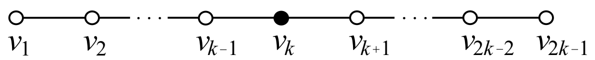

Figure 4: The path P_{2k-1} with v_k as the only kPVC-perfect vertex

The definition of a q-kPVC-imperfect vertex gives the desired theorems similar to Theorem 2.3.

Theorem 2.11. *Let G and H be connected graphs, where G is different from the vertex graph, and graph H rooted at $h \in V(H)$. If h is a q-$kPVC$-imperfect vertex for some integer $q \geq \left\lceil \frac{k}{2} \right\rceil$, then*

$$\psi_k(G \circ H) = |V(G)|\psi_k(H) + \psi_2(G)\,.$$

Proof. Let h b a q-kPVC-imperfect vertex for some $q \geq \left\lceil \frac{k}{2} \right\rceil$. Without loss of generality, suppose that the root vertex h is the vertex h_1. Let S be a minimum k-path vertex cover of graph $G \circ H$, $S_i = \{(g_i, h_j) \in S \,|\, j \in \{2, \ldots, |V(H)|\}\}$, $i \in \{1, \ldots, |V(G)|\}$, and $S' = \{(g_i, h_1) \in S \,|\, i \in \{1, \ldots, |V(G)|\}\}$. It is obvious that

$$|S| = \left(\sum_{i=1}^{|V(G)|} |S_i| \right) + |S'|\,.$$

By the definition of the kPVC-imperfect vertex h_1 always lies in an uncovered path of order at least q for every minimum k-path vertex cover of graph H in such a away that h_1 is an end-vertex of this path. Since h_1 is a kPVC-imperfect vertex, every minimum k-path vertex cover of H does not contain vertex h_1. If $|S_i| = \psi_k(H) - 1$, for some i, then $S_i \cup \{(g_i, h_1)\}$ is a minimum k-path vertex cover of graph induced by the H_i-layer, and hence h_1 is a kPVC-perfect vertex, which is not possible. Therefore, $|S_i| \geq \psi_k(H)$, for all $i \in \{1, \ldots, |V(G)|\}$.

The main idea of the proof is to show that for any edge in the G-layer at least one of its end-vertices in S. Hence, any two adjacent H-layers contribute at least $2\psi_k(H) + 1$ vertices to S. Let $(g_i, h_1), (g_j, h_1) \in V(G \circ H)$, $i \neq j$, be any two adjacent vertices. We analyze two cases.

Case 1: Let $|S_i| = \psi_k(H)$ and $|S_j| = \psi_k(H)$. Suppose that both vertices (g_i, h_1) and (g_j, h_1) do not belong to S. Then S_i and S_j are minimum k-path vertex covers of the H_i-layer and the H_j-layer, respectively. If h_1 is a q-kPVC-imperfect vertex of graph H, then (g_i, h_1) and (g_j, h_1) are q-kPVC-imperfect vertices of the H_i-layer and the H_j-layer, respectively. Hence, (g_i, h_1) lies in an uncovered path P_r, $r \geq q$, and is an end-vertex of this path. Also, (g_j, h_1) lies in an uncovered path P_s, $s \geq q$, and is an end-vertex of this path. Since vertices (g_i, h_1) and (g_j, h_1) are adjacent in graph $G \circ H$, paths P_r and P_s together form another uncovered path of order

$$r + s \geq 2 \cdot q \geq 2 \cdot \left\lceil \frac{k}{2} \right\rceil \geq k\,.$$

Hence, S is not a k-path vertex cover, which is a contradiction. Therefore, at least one of the vertices (g_i, h_1) and (g_j, h_1) must belong to S. Moreover, layers H_i and H_j contribute at least $2\psi_k(H) + 1$ vertices to S.

Case 2: At least one of $|S_i|$ and $|S_j|$ does not equal $\psi_k(H)$. Without loss of generality, let this be $|S_i|$. According to the observation above $|S_i| \geq \psi_k(H)+1$ and $|S_j| \geq \psi_k(H)$. Obviously, layers H_i and H_j contribute at least $2\psi_k(H) + 1$ vertices to S.

Considering both cases,

$$|S| \geq |V(H)|\psi_k(H) + \psi_2(G)\,.$$

By Proposition 2.1, this is also the upper bound. Hence,

$$\psi_k(G \circ H) = |V(H)|\psi_k(H) + \psi_2(G)\,.$$

□

Theorem 2.12. *Let G and H be connected graphs, where G is different from the vertex graph, and graph H rooted at $h \in V(H)$. If*

$$\psi_k(G \circ H) = |V(G)|\psi_k(H) + \psi_2(G)\,,$$

then h is a q-kPVC-imperfect vertex for some integer $q \geq \left\lfloor \frac{k}{2} \right\rfloor$.

Proof. First note that $\psi_2(G) \neq 0$ since G is connected and different from the vertex graph. Suppose that h is a q-kPVC-imperfect vertex for some $q \leq \left\lfloor \frac{k}{2} \right\rfloor - 1$. Note, if $k = 2$ or $k = 3$, then $q = 0$ and h is a kPVC-perfect vertex. By Remark 2.4 it follows that

$$\psi_k(G \circ H) = |V(G)|\psi_k(H) < |V(G)|\psi_k(H) + \psi_2(G)\,.$$

Let $k \geq 4$. We may assume that $q \neq 0$ (otherwise the proof is the same as above). First, we construct a k-path vertex cover S of graph $G \circ H$ such that $|S| = |V(G)|\psi_k(H) + \psi_2(G)$. Let S_1 be a minimum k-path vertex cover of graph H, such that h is an end-vertex of an uncovered path of order q, and S_2 a minimum 2-path vertex cover (i.e. vertex cover) of graph G. We cover every H-layer in the same way as S_1 covers graph H. Also, we cover the G-layer in the same way as S_2 covers graph G. We take both mentioned covers for the set S. Note that $S_2 \neq \emptyset$ since G is connected and different from the vertex graph. Take a vertex $(g, h) \in V(G \circ H)$ such that $g \in S_2$. Let T_2

be the set of vertices in graph G which are adjacent to g and do not belong to S_2. Since S_2 is a minimum vertex cover $T_2 \neq \emptyset$. The graph induced on the set of vertices $T_2 \cup \{g\}$ is a star graph with the central vertex g. Vertices in $V(G) - (T_2 \cup \{g\})$ (if there are any) that are adjacent to vertices in T_2 must all belong to S_2. Otherwise, S_2 would not be a vertex cover. Therefore, by uncovering the vertex g, we get an uncovered path of order at most 3 in graph G. For $|T_2| = 1$ and $u_i \in T_2$, this path is of order 2, namely $P_2 = u_i g$. The worst case is if $|T_2| \geq 2$. For vertices $u_i, u_j \in T_2$, $i \neq j$, this path is of order 3, namely $P_3 = u_i g u_j$. It is obvious that if we eliminate paths of order k in the case of $|T_2| \geq 2$, we also eliminate them in the case of $|T_2| = 1$. Hence, we consider two vertices $u_i, u_j \in T_2$, $i \neq j$.

If h is a q-kPVC-imperfect vertex of graph H, then (g_i, h) and (g_j, h) are q-kPVC-imperfect vertices of the H_i-layer and the H_j-layer, respectively. Hence, (g_i, h) lies in an uncovered path P_q and is an end-vertex of this path. Also, (g_j, h) lies in an uncovered path P_q and is an end-vertex of this path. Since vertices (g_i, h), (g, h) and (g_j, h) form the path P_3, both paths P_q together with the path P_3 form another uncovered path of order at most

$$2 \cdot q + 1 \leq 2 \cdot \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) + 1 \leq k - 1 \,.$$

We have proved that $S - \{(g, h)\}$ is also a k-path vertex cover of graph $G \circ H$. Therefore

$$\psi_k(G \circ H) \leq |V(G)| \psi_k(H) + \psi_2(G) - 1 < |V(G)| \psi_k(H) + \psi_2(G) \,.$$

□

For even k we can combine Theorem 2.11 and Theorem 2.12 into the following corollary.

Corollary 2.13. *Let G and H be connected graphs, where G is different from the vertex graph, and graph H rooted at $h \in V(H)$. If k is even, then*

$$\psi_k(G \circ H) = |V(G)| \psi_k(H) + \psi_2(G)$$

if and only if h is a q-kPVC-imperfect vertex for some integer $q \geq \frac{k}{2}$.

Proof. For k is even the equality $\left\lceil \frac{k}{2} \right\rceil = \left\lfloor \frac{k}{2} \right\rfloor$ holds. Hence, Theorem 2.12 is the converse of Theorem 2.11 and vise versa. □

To see the behavior of $\psi_k(G \circ H)$ for smaller values of q for a q-kPVC-imperfect vertex we give the following result.

Proposition 2.14. *Let G and H be connected graphs, where G is different from the vertex graph, and graph H rooted at $h \in V(H)$. If h is a 1-kPVC-imperfect vertex, then*

$$\psi_k(G \circ H) = |V(G)|\psi_k(H) + \psi_k(G)\,.$$

Proof. Let $h \in V(H)$ be a 1-kPVC-imperfect vertex. Then there exists a minimum k-path vertex cover S of graph H such that h is uncovered and isolated from the other uncovered vertices in H. We construct a k-path vertex cover of graph $G \circ H$ in such way that we cover every H-layer in the same way as S covers graph H. In this sense vertices (g_i, h), $i \in \{1, \ldots, |V(G)|\}$, are all uncovered and isolated from the other uncovered vertices in all H-layers. To complete the construction we cover the vertices of the G-layer with a k-path vertex cover of the size $\psi_k(G)$. Altogether we have covered $|V(H)|\psi_k(H) + \psi_k(G)$ vertices and since, according to Proposition 2.5, this is also the lower bound for $\psi_k(G \circ H)$, it follows that

$$\psi_k(G \circ H) = |V(G)|\psi_k(H) + \psi_k(G)\,.$$

□

The converse of Proposition 2.14 does not hold. Take for example $k \geq 5$, $G = P_{k-3} = u_1u_2\ldots u_{k-3}$, and $H = P_{k+2} = v_1v_2\ldots v_{k+2}$ rooted at v_1. It is clear that $\psi_k(H) = 1$ and that v_1 is a 2-kPVC-imperfect vertex since the closest vertex to v_1 which can be covered in a minimum k-path vertex cover of $H = P_{k+2}$ is vertex v_3. It is easy to see that $\psi_k(G \circ H) = |V(H)|\psi_k(H) + \psi_k(G)$ (see Figure 5).

Corollary 2.15. *Let G and H be arbitrary connected graphs and H rooted at $h \in V(H)$. Then*

$$\psi_3(G \circ H) = \begin{cases} |V(G)|\psi_3(H) & ;\ h \textit{ is a } k\textit{PVC-perfect vertex} \\ |V(G)|\psi_3(H) + \psi_3(G) & ;\ h \textit{ is a } 1\text{-}k\textit{PVC-imperfect vertex} \\ |V(G)|\psi_3(H) + \psi_2(G) & ;\ h \textit{ is a } 2\text{-}k\textit{PVC-imperfect vertex} \end{cases}.$$

Proof. If G is the vertex graph, then $\psi_2(G) = \psi_3(G) = 0$. It follows that

$$\begin{aligned}\psi_3(G \circ H) = \psi_3(H) &= |V(G)|\psi_3(H) \\ &= |V(G)|\psi_3(H) + \psi_3(G) \\ &= |V(G)|\psi_3(H) + \psi_2(G)\end{aligned}$$

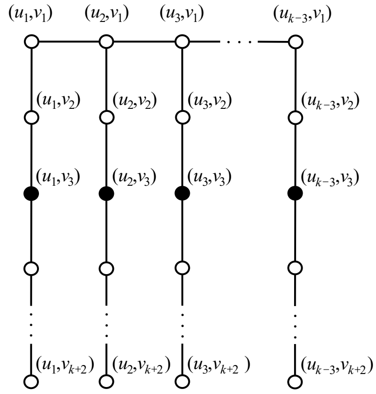

Figure 5: The graph $P_{k-3} \circ P_{k+2}$ rooted at (u_i, v_1), $i \in \{1, \ldots, k-1\}$

and all three results coincide.

Now suppose that G is not the vertex graph. If h is a kPVC perfect vertex, then by Remark 2.4 it follows that

$$\psi_k(G \circ H) = |V(G)|\psi_k(H)\,.$$

Let $q \geq \lceil \frac{k}{2} \rceil = \lceil \frac{3}{2} \rceil = 2$. If h is a q-kPVC-imperfect vertex, and hence also a 2-kPVC-imperfect vertex, then by Theorem 2.11 it follows that

$$\psi_3(G \circ H) = |V(G)|\psi_3(H) + \psi_2(G)\,.$$

To end the proof, by Proposition 2.14, if h is a 1-kPVC-imperfect vertex, then

$$\psi_3(G \circ H) = |V(G)|\psi_3(H) + \psi_3(G)\,.$$

□

Corollary 2.16. *Let G and H be arbitrary connected graphs and H rooted at $h \in V(H)$. Then*

$$\mathrm{diss}(G \circ H) = \begin{cases} |V(G)|\mathrm{diss}(H) & ;\ h \text{ is a } kPVC\text{-perfect vertex} \\ |V(G)|(\mathrm{diss}(H) - 1) + \mathrm{diss}(G) & ;\ h \text{ is a } 1\text{-}kPVC\text{-imperfect vertex} \\ |V(G)|(\mathrm{diss}(H) - 1) + \alpha(G) & ;\ h \text{ is a } 2\text{-}kPVC\text{-imperfect vertex} \end{cases}.$$

Proof. By Corollary 2.15 the result follows immediately. First, suppose that

h is a kPVC-perfect vertex. Then

$$\begin{aligned}\mathrm{diss}(G \circ H) &= |V(G \circ H)| - \psi_3(G \circ H)\\ &= |V(G)||V(H)| - |V(G)|\psi_3(H)\\ &= |V(G)|(|V(H)| - \psi_3(H))\\ &= |V(G)|\mathrm{diss}(H)\,.\end{aligned}$$

If h is a 1-PVC-imperfect vertex, then

$$\begin{aligned}\mathrm{diss}(G \circ H) &= |V(G \circ H)| - \psi_3(G \circ H)\\ &= |V(G)||V(H)| - |V(G)|\psi_3(H) - \psi_3(G)\\ &= |V(G)||V(H)| - |V(G)|\psi_3(H) - |V(G)| + |V(G)| - \psi_3(G)\\ &= |V(G)|(|V(H)| - \psi_3(H) - 1) + |V(G)| - \psi_3(G)\\ &= |V(G)|(\mathrm{diss}(H) - 1) + \mathrm{diss}(G)\,.\end{aligned}$$

If h is a 2-PVC-imperfect vertex, then

$$\begin{aligned}\mathrm{diss}(G \circ H) &= |V(G \circ H)| - \psi_3(G \circ H)\\ &= |V(G)||V(H)| - |V(G)|\psi_3(H) - \psi_2(G)\\ &= |V(G)||V(H)| - |V(G)|\psi_3(H) - |V(G)| + |V(G)| - \psi_2(G)\\ &= |V(G)|(|V(H)| - \psi_3(H) - 1) + |V(G)| - \psi_2(G)\\ &= |V(G)|(\mathrm{diss}(H) - 1) + \alpha(G)\,.\end{aligned}$$

□

3. Concluding remarks

We have seen that securing local networks which are communicating with each other through servers that are connected in a global network can be done in such a way that we place a server in a kPVC-perfect vertex of a local network. In this sense we get a secured network with the smallest possible number of sensors. If this is not possible, then the server must be placed as close as possible to a kPVC-perfect vertex in the local networks.

This study was made in the case where all local networks are the same. In general, local networks are different. Hence, the study of generalized rooted product of graphs is needed. This product was introduced in [8]. Let G be a labeled graph on m vertices and let $\mathcal{H}$ be a sequence of m rooted graphs

$H_1, H_2, \ldots, H_m$. The rooted product graph $G(\mathcal{H})$ is the graph obtained by identifying the root of graph H_i with the i-th vertex of graph G.

We end this short section with an open question of how to properly secure a generalized rooted product.

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