The partition dimension of strong product graphs and Cartesian product graphs

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Abstract

Let $G = (V, E)$ be a connected graph. The distance between two vertices $u, v \in V$, denoted by $d(u, v)$, is the length of a shortest u, v-path in G. The distance between a vertex $v \in V$ and a subset $P \subset V$ is defined as $\min\{d(v, x) : x \in P\}$, and it is denoted by $d(v, P)$. An ordered partition $\{P_1, P_2, ..., P_t\}$ of vertices of a graph G, is a resolving partition of G, if all the distance vectors $(d(v, P_1), d(v, P_2), ..., d(v, P_t))$ are different. The partition dimension of G is the minimum number of sets in any resolving partition of G. In this article we study the partition dimension of strong product graphs and Cartesian product graphs. Specifically, we prove that the partition dimension of the strong product of graphs is bounded below by four and above by the product of the partition dimensions of the factor graphs. Also, we give the exact value of the partition dimension of strong product graphs when one factor is a complete graph and the other one is a path or a cycle. For the case of Cartesian product graphs, we show that its partition dimension is less than or equal to the sum of the partition dimensions of the factor graphs minus one. Moreover, we obtain an upper bound on the partition dimension of Cartesian product graphs, when one factor is a complete graph.

Keywords: Resolving partition; partition dimension; strong product graphs; Cartesian product graphs; graphs partitioning.

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1 Introduction

The idea of a partition dimension was introduced by Chartrand *et al.* in [6] to gain more insight about another closely related graph parameter called the metric dimension of a graph. The partition dimension of graphs is also studied in [3, 7, 16, 17]. Given a connected graph $G = (V, E)$ and an ordered partition $\Pi = \{P_1, P_2, ..., P_t\}$ of the vertices of G, the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), ..., d(v, P_t))$, where $d(v, P_i)$, with $1 \le i \le t$, represents the distance between the vertex v and the set P_i, that is $d(v, P_i) = \min_{u \in P_i}\{d(v, u)\}$ ($d(v, u)$ denotes the distance between the vertices v and u). We say that Π is a *resolving partition* of G if for every pair of distinct vertices $u, v \in V$, $r(u|\Pi) \ne r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition of G and is denoted by $pd(G)$.

The concepts of resolvability and location in graphs were described independently by Harary and Melter [8], and Slater [15], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic [1, 2, 3, 4, 5, 6, 7, 12]. Slater described the usefulness of these ideas into long range aids to navigation [15]. Also, these concepts have some applications in chemistry for representing chemical compounds [10, 11] or in problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [13]. Other applications of this concept to navigation of robots in networks and other areas appear in [4, 9, 12].

Given a connected graph $G = (V, E)$ and an ordered set of vertices $S = \{v_1, v_2, ..., v_k\}$ of G, the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), ..., d(v, v_k))$. We say that S is a *resolving set* of G if for every pair of distinct vertices $u, v \in V$, $r(u|S) \ne r(v|S)$. The *metric dimension* of G is the minimum cardinality of any resolving set of G, and it is denoted by $dim(G)$. The metric dimension of graphs is studied in [1, 2, 3, 4, 5, 16].

It is natural to think that the partition dimension and metric dimension are related; in [6] it was shown that for any nontrivial connected graph G we have

$$pd(G) \le dim(G) + 1. \tag{1}$$

We recall that the strong product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G \boxtimes H = (V, E)$, such that $V = \{(a, b) \ : \ a \in V_1, \ b \in V_2\}$ and two vertices $(a, b) \in V$ and $(c, d) \in V$ are adjacent in $G \boxtimes H$ if and only if, either

- $a = c$ and $bd \in E_2$, or
- $b = d$ and $ac \in E_1$, or
- $ac \in E_1$ and $bd \in E_2$.

Also, the Cartesian product of G and H is the graph $G \Box H = (V, E)$, such that $V = \{(a, b) \ : \ a \in V_1, \ b \in V_2\}$ and two vertices $(a, b) \in V$ and $(c, d) \in V$ are adjacent in $G \boxtimes H$ if and only if, either

- $a = c$ and $bd \in E_2$, or
- $b = d$ and $ac \in E_1$.

Let $v \in V_2$. We refer to the set $V_1 \times \{v\}$ as a G-layer. Similarly $\{u\} \times V_2$, $u \in V_1$ is an H-layer. When referring to a specific G or H layer, we denote them by G^v or ${}^uH$, respectively. Layers can also be regarded as the graphs induced on these sets. Obviously, a G-layer or H-layer is isomorphic to G or H, respectively.

Studies about partition dimension in product graphs have been presented in [17], where the authors obtained some results about the partition dimension of the Cartesian product graphs. Also in [14] several results about the partition dimension of corona product graphs were presented. In this article we begin with the study of the partition dimension of strong product graphs and we also continue with the study of the partition dimension of Cartesian product graphs.

2 Strong product graphs

We begin with the following useful lemmas.

Lemma 1. *Let G and H be two connected graphs and let A and B be two proper subsets of vertices of G and H, respectively. If $a \in A$ and $b \notin B$, then*

$$d_{G\boxtimes H}((a,b), A \times B) = \min_{v\in B}\{d_H(b,v)\}.$$

Equivalently, if $a \notin A$ and $b \in B$, then

$$d_{G\boxtimes H}((a,b), A \times B) = \min_{u\in A}\{d_G(a,u)\}.$$

Proof. Suppose $a \in A$ and $b \notin B$. We first prove that for every $v \in B$, $d_{G\boxtimes H}((a,b), A \times \{v\}) = d_H(b,v)$.

$$\begin{aligned}
d_{G\boxtimes H}((a,b), A \times \{v\}) &= \min_{(u,v)\in A\times\{v\}}\{d_{G\boxtimes H}((a,b),(u,v))\} \\
&= \min_{u\in A}\{d_{G\boxtimes H}((a,b),(u,v))\} \\
&= \min_{u\in A}\{\max\{d_G(a,u), d_H(b,v)\}\} \\
&= \min_{u\in A}\{\max_{a=u}\{d_G(a,u), d_H(b,v)\}, \max_{a\neq u}\{d_G(a,u), d_H(b,v)\}\} \\
&= \min_{u\in A}\{d_H(b,v), \max_{a\neq u}\{d_G(a,u), d_H(b,v)\}\} \\
&= d_H(b,v).
\end{aligned}$$

Thus we obtain that $d_{G\boxtimes H}((a,b), A\times B) = \min_{v\in B}\{d_{G\boxtimes H}((a,b), A\times\{v\})\} = \min_{v\in B}\{d_H(b,v)\}$. Analogously we prove that if $a \notin A$ and $b \in B$, then $d_{G\boxtimes H}((a,b), A\times B) = \min_{u\in A}\{d_G(a,u)\}$ and the proof is complete. □

Lemma 2. [6] *Let G be a connected graph of order $n \geq 2$. Then $pd(G) = 2$ if and only if G is a path graph.*

The following straightforward claim is useful to prove our next results.

Claim 3. *Let G and H be two connected non-trivial graphs. If there exists a resolving partition of $G \boxtimes H$ with exactly three sets, say $\Pi = \{A, B, C\}$, then there exists no subgraph of $G \boxtimes H$ isomorphic to K_4, such that it contains at least one vertex from each of the sets A, B and C.*

Lemma 4. *Let G and H be two connected non-trivial graphs. If there exists a resolving partition of $G \boxtimes H$ with exactly three sets, say $\Pi = \{A, B, C\}$, then there exists no subgraph of $G \boxtimes H$ isomorphic to K_4, such that exactly three of its vertices belong to exactly one of the sets A, B, C.*

Proof. Suppose there is a subgraph of $G \boxtimes H$ isomorphic to K_4 with exactly three vertices in A, say x_1, x_2 and x_3. Assume the fourth vertex is in B. Let z_1, z_2 and z_3 be vertices from C closest to x_1, x_2 and x_3, respectively. Let $d_{G\boxtimes H}(x_1, z_1) = a$. Then $d_{G\boxtimes H}(x_2, z_2) \leq a + 1$, otherwise z_1 would be closer to x_2 than z_2. Also, $d_{G\boxtimes H}(x_2, z_2) \geq a - 1$, otherwise z_2 would be closer to x_1 than z_1. Similarly, $a - 1 \leq d_{G\boxtimes H}(x_3, z_3) \leq a + 1$. Since Π is a resolving partition and x_1, x_2 and x_3 are mutually adjacent and all adjacent to a vertex from B, the distances between x_2 and z_2 (x_3 and z_3) cannot be equal to a.

Assume now, $d_{G\boxtimes H}(x_2, z_2) \neq d_{G\boxtimes H}(x_3, z_3)$, moreover we can choose the notation in such way that $d_{G\boxtimes H}(x_2, z_2) = a + 1$ and $d_{G\boxtimes H}(x_3, z_3) = a - 1$. But $d_{G\boxtimes H}(x_2, z_3) \leq d_{G\boxtimes H}(x_2, x_3) + d_{G\boxtimes H}(x_3, z_3) \leq a$. So z_3 is closer to x_2 than z_2, a contradiction.

It follows that at least two of the vertices x_1, x_2 and x_3 have the same distance to C, a contradiction, since Π is a resolving partition. □

Theorem 5. *For any connected non trivial graphs G and H,*

$$4 \leq pd(G \boxtimes H) \leq pd(G) \cdot pd(H).$$

Proof. Let $\Pi_1 = \{A_1, A_2, ..., A_k\}$ and $\Pi_2 = \{B_1, B_2, ..., B_t\}$ be resolving partitions of $G = (V_1, E_1)$ and $H = (V_2, E_2)$ respectively. Let us show that $\Pi = \{A_i \times B_j : 1 \leq i \leq k, 1 \leq j \leq t\}$ is a resolving partition of $G \boxtimes H$. Let (a, b), (c, d) be two different vertices of $V_1 \times V_2$. Notice that if $(a, b), (c, d)$ belong to different sets in Π, then they are resolved by Π. So, we suppose that $(a, b), (c, d)$ belong to the same set in Π.

If $a = c$, then let $i \in \{1, ..., k\}$ such that $a \in A_i$. Hence there exists $B_j \in \Pi_2$, for some $j \in \{1, \ldots, t\}$, such that $d_H(b, B_j) \neq d_H(d, B_j)$. Now according to the Lemma 1 we have that $d_{G\boxtimes H}((a, b), A_i \times B_j) = \min_{v\in B_j}\{d_H(b, v)\}$ and $d_{G\boxtimes H}((c, d), A_i \times B_j) = \min_{v\in B_j}\{d_H(d, v)\}$. Thus we obtain

$$\begin{aligned}
d_{G\boxtimes H}((a, b), A_i \times B_j) &= \min_{v\in B_j}\{d_H(b, v)\} \\
&= d_H(b, B_j) \\
&\neq d_H(d, B_j) \\
&= \min_{v\in B_j}\{d_H(d, v)\} \\
&= d_{G\boxtimes H}((c, d), A_i \times B_j).
\end{aligned}$$

On the contrary, if $a \neq c$, then there exists $A_j \in \Pi_1$, for some $j \in \{1, \ldots, k\}$, such that $d_G(a, A_j) \neq d_G(c, A_j)$. Also, there exists $l \in \{1, ..., t\}$ such that $b, d \in B_l$. Now, by using Lemma 1 and proceeding as above we obtain that $d_{G\boxtimes H}((a, b), A_j \times B_l) \neq d_{G\boxtimes H}((c, d), A_j \times B_l)$. Therefore for every two different vertices $(a, b), (c, d) \in V_1 \times V_2$, we have that $r((a, b)|\Pi) \neq r((c, d)|\Pi)$ and the upper bound follows.

On the other hand, by Lemma 2 we have that $pd(G \boxtimes H) \geq 3$. Suppose, $pd(G \boxtimes H) = 3$. Let $\Pi = \{A, B, C\}$ be a resolving partition for $G \boxtimes H$. Let $V_1 = \{u_1, ..., u_t\}$ and $V_2 = \{v_1, ..., v_r\}$ be the vertex sets of G and H, respectively.

Suppose, for every $i \in \{1, ..., t\}$ and $j \in \{1, ..., r\}$ the layers ${}^{u_i}H$ and G^{v_j} have a non-empty intersection with each of A, B and C. Let $v_j v_{j'}$ be an edge in H, and $u_i u_{i'}$ an edge in G. Using Claim 3 the vertices $(u_i, v_j), (u_i, v_{j'}), (u_{i'}, v_j)$ and $(u_{i'}, v_{j'})$ belong to at most two different sets from Π, say A and B. Moreover, from Lemma 4, either all belong to the same set (A or B), or two belong to A, and the other two belong to B. Consider the following cases.

Case 1: Let $(u_i, v_j), (u_{i'}, v_{j'}) \in A$, and $(u_i, v_{j'}), (u_{i'}, v_j) \in B$. Since ${}^{u_i}H \cap C \neq \emptyset$ and ${}^{u_{i'}}H \cap C \neq \emptyset$, there exists a vertex $v_k \in V_2$, such that (u_i, v_k) and $(u_{i'}, v_k)$ both belong to the same set (either A or B), and $(u_i, v_{k'})$ and $(u_{i'}, v_{k'})$ belong to two different sets (A or B), where $v_{k'}$ is a neighbor of v_k. This is a contradiction to Lemma 4.

Case 2: Let $(u_i, v_j), (u_i, v_{j'}) \in A$, and $(u_{i'}, v_{j'}), (u_{i'}, v_j) \in B$. This also leads to a contradiction, similarly to Case 1.

Case 3: Let $(u_i, v_j), (u_{i'}, v_j) \in A$, and $(u_i, v_{j'}), (u_{i'}, v_{j'}) \in B$. This also leads to a contradiction, similarly to Case 1, instead of considering H-layers, we consider G-layers.

Case 4: Let $(u_i, v_j), (u_i, v_{j'}), (u_{i'}, v_{j'}), (u_{i'}, v_j) \in A$. We can reduce this case to one of the previous cases. Also, a contradiction.

It follows that there exists at least one G-layer or H-layer which has an empty intersection with at least one of A, B and C, say A. Since G and H are both connected and non-trivial, for every $u_i \in V(G)$ and $v_j \in V(H)$ there exists at least one neighbor. We use the notation $u_{i'}$ or $v_{j'}$ to refer to such a neighbor, respectively. Without loss of generality, suppose the layer ${}^{u_i}H$ has empty intersection with A, where i is chosen is such way that ${}^{u_{i'}}H$ contains at least one vertex from A (such an i exists since Π is a partition of order three). Let $(u_{i'}, v_j) \in A$ and let $(u_i, v_j) \in B$. By Claim 3 and Lemma 4 the vertex $(u_{i'}, v_{j'})$ is also in A, moreover $(u_i, v_{j'})$ is also in B. Using the same argument, all vertices in ${}^{u_{i'}}H$ are in A and all vertices in ${}^{u_i}H$ are in B.

Since Π is a partition of order three, there exists a vertex from C in graph $G \boxtimes H$. Let $w = (u_l, v_k)$ be such a vertex closest to ${}^{u_{i'}}H$. So the vertices $(u_{i'}, v_k), (u_{i'}, v_{k'})$ have the same partition representation, which is a contradiction with Π being a resolving partition. Therefore, $pd(G \boxtimes H) > 3$. This completes the proof of the lower bound. $\square$

The above bounds are tight. For instance, the upper bound is attained for the strong product graph $K_r \boxtimes K_t$. Also for the grid graph $P_t \boxtimes P_r$ we observe that both bounds are achieved.

Corollary 6. *For integers $r, t \geq 2$, $pd(P_t \boxtimes P_r) = 4$.*

Even though the bounds of Theorem 5 are tight we can find examples in which such bounds are not achieved, as we will show in the following results. Next we obtain the partition dimension for some specific families of strong product graphs. First we present a remark which we will use in some proofs.

Remark 7. *Let G be any connected graph. If $(a, b), (a, c) \in {}^{a}K_n$ are two different vertices of $G \boxtimes K_n$, then $d_{G\boxtimes K_n}((a, b), (x, y)) = d_{G\boxtimes K_n}((a, c), (x, y))$ for every $(x, y) \notin \{(a, b), (a, c)\}$ of $G \boxtimes K_n$.*

Theorem 8. *For any connected non-complete graph G of order $t \geq 3$ and any integer $n \geq 2$,*

$$pd(G \boxtimes K_n) \geq n + 2.$$

Proof. Let $V_1 = \{u_1, ..., u_t\}$ and $V_2 = \{v_1, ..., v_n\}$ be the vertex sets of G and K_n, respectively. From Remark 7, we have that for every vertex $u_i \in V_1$ and every pair of distinct vertices $v_j, v_l \in V_2$, it follows that (u_i, v_j) and (u_i, v_l) belong to different sets in any resolving partition of $G \boxtimes K_n$. So $pd(G \boxtimes K_n) \geq n$. Let Π be a resolving partition of minimum cardinality in $G \boxtimes K_n$. If $pd(G \boxtimes K_n) = n$ or $pd(G \boxtimes K_n) = n+1$, then there exist two adjacent vertices $u_l, u_k \in V_1$, $l, k \in \{1, ..., t\}$, such that for every $A \in \Pi$ it follows that $A \cap (\{u_l, u_k\} \times V_2) \neq \emptyset$. Moreover, there exists at least a set $B \in \Pi$ and two vertices $(u_l, v_i), (u_k, v_j) \in (\{u_l, u_k\} \times V_2)$ such that $(u_l, v_i), (u_k, v_j) \in B$. So we obtain that $d_{G \boxtimes K_n}((u_l, v_i), C) = d_{G \boxtimes K_n}((u_k, v_j), C)$ for every $C \in \Pi$, $C \neq B$, which is a contradiction. Thus $pd(G \boxtimes K_n) \geq n+2$ and the proof is complete. □

Proposition 9. *For integers $n, t \geq 2$,*

$$pd(P_t \boxtimes K_n) = \begin{cases} 2n, & \text{if } t = 2, \\ n+2, & \text{if } t \geq 3. \end{cases}$$

Proof. Let $V_1 = \{u_1, ..., u_t\}$ and $V_2 = \{v_1, ..., v_n\}$ be the vertex sets of P_t and K_n, respectively. We assume $u_i u_{i+1} \in E(P_t)$, for all $i \in \{1, \ldots, t-1\}$. If $t = 2$, then $pd(P_t \boxtimes K_n) = pd(K_2 \boxtimes K_n) = pd(K_{2n}) = 2n$. Thus, from now on we suppose $t \geq 3$. By Theorem 8 we have that $pd(P_t \boxtimes K_n) \geq n+2$. On the other hand, let $\Pi = \{A_1, A_2, ..., A_{n+2}\}$ be the vertex partition, where

$$\begin{aligned} A_1 &= \{(u_1, v_1)\}, \\ A_2 &= \{(u_2, v_1)\}, \\ A_3 &= \{(u_3, v_1), (u_4, v_1), ..., (u_t, v_1)\}, \\ A_4 &= \{(u_1, v_2), (u_2, v_2), ..., (u_t, v_2)\}, \\ A_5 &= \{(u_1, v_3), (u_2, v_3), ..., (u_t, v_3)\}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{n+2} &= \{(u_1, v_n), (u_2, v_n), ..., (u_t, v_n)\}. \end{aligned}$$

Figure 1 shows the partition for the case of $P_6 \boxtimes K_4$.

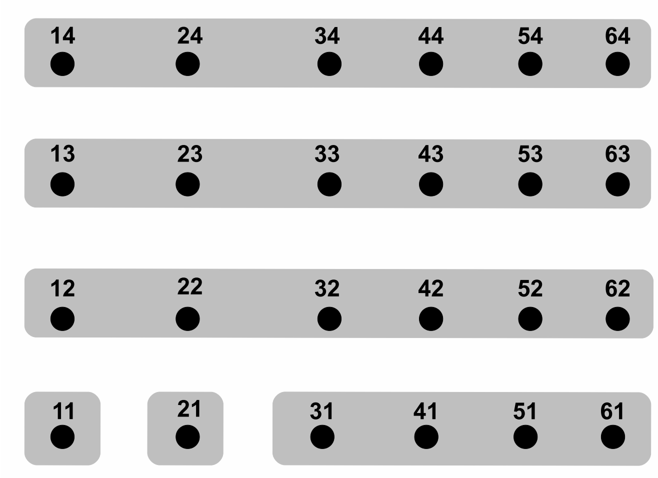

Figure 1: The partition Π of $P_6 \boxtimes K_4$. Vertex labeled by ij represents the vertex (u_i, v_j). Edges of the graph have not been drawn.

We shall prove that Π is a resolving partition for $P_t \boxtimes K_n$. We consider two different vertices $(u_i, v_j), (u_l, v_k) \in V_1 \times V_2$ belonging to a same set of the partition Π. Thus, $j = k$. We can suppose without loss of generality that $i < l$. If $i, l \geq 3$, then we have that $d_{P_t \boxtimes K_n}((u_i, v_j), A_1) = i - 1 \neq l - 1 = d_{P_t \boxtimes K_n}((u_l, v_k), A_1)$. If $i < 3$ and $l \geq 3$, then $i \in \{1, 2\}$ and we have that $d_{P_t \boxtimes K_n}((u_i, v_j), A_1) = 1 < l - 1 = d_{P_t \boxtimes K_n}((u_l, v_k), A_1)$. Finally, if $i, l < 3$, then $i = 1$ and $l = 2$. So we have $d_{P_t \boxtimes K_n}((u_i, v_j), A_3) = 2 > 1 = d_{P_t \boxtimes K_n}((u_l, v_k), A_3)$. Thus, Π is a resolving partition for $P_t \boxtimes K_n$ and $pd(P_t \boxtimes K_n) \leq n + 2$. Therefore the result follows. $\square$

Proposition 10. *For integers $n, t \geq 3$,*

$$pd(C_t \boxtimes K_n) = \begin{cases} 3n, & \text{if } t = 3, \\ n+3, & \text{if } t = 4 \text{ or } t = 5, \\ n+2, & \text{if } t \geq 6. \end{cases}$$

Proof. Let $V_1 = \{u_0, ..., u_{t-1}\}$ and $V_2 = \{v_1, ..., v_n\}$ be the vertex sets of C_t and K_n, respectively. We assume $u_i u_{i+1} \in E(C_t)$, for all $i \in \{0, \ldots, t-1\}$. In this proof, operations with the indices of u_i are done modulo t. If $t = 3$, then $pd(C_t \boxtimes K_n) = pd(K_t \boxtimes K_n) = pd(K_{tn}) = tn$. We may assume that $t \geq 4$.

By Theorem 8 we have that $pd(C_t \boxtimes K_n) \geq n + 2$. We first analyze the case $t = 4$. We suppose that $pd(C_4 \boxtimes K_n) = n + 2$ and let $\{A_1, A_2, ..., A_{n+2}\}$ be a resolving partition for $C_4 \boxtimes K_n$. According to this partition, we say that if $v \in A_i$, then v has label i.

We consider a set ${}^{u_i}K_n$. From Remark 7 we have that every two different vertices of ${}^{u_i}K_n$ have different labels. Hence, without loss of generality (w.l.g.) we suppose that the n vertices of ${}^{u_i}K_n$ are labeled from 1 to n. Since there are only $n + 2$ possible labels for the sets ${}^{u_{i-1}}K_n$ and ${}^{u_{i+1}}K_n$, we have that at least $n - 2$ labels of the set $\{1, ..., n\}$ are repeated in the labeling of ${}^{u_{i-1}}K_n$ and ${}^{u_{i+1}}K_n$. If ${}^{u_{i-1}}K_n$ or ${}^{u_{i+1}}K_n$ contains the labels $n + 1$ and $n + 2$ (for instance ${}^{u_{i-1}}K_n$), then there exist two vertices $u \in {}^{u_{i-1}}K_n$ and $v \in {}^{u_i}K_n$ having the same label and also, they have the same distance (1) to every other label, which is a contradiction. Thus, for the sets ${}^{u_{i-1}}K_n$ and ${}^{u_{i+1}}K_n$ we have that at least $n - 1$ labels of the set $\{1, ..., n\}$ are repeated in the labeling of ${}^{u_{i-1}}K_n$ and ${}^{u_{i+1}}K_n$. Moreover, at least $n - 2$ of these labels $\{1, ..., n\}$ are the same in ${}^{u_i}K_n$, ${}^{u_{i-1}}K_n$ and ${}^{u_{i+1}}K_n$.

Now, if ${}^{u_{i-1}}K_n$ is labeled from 1 to n, then w.l.g. we can suppose that ${}^{u_{i+1}}K_n$ contains the label $n + 1$ and ${}^{u_{i+2}}K_n$ contains the label $n + 2$. Thus, there exists two vertices $u \in {}^{u_{i-1}}K_n$ and $v \in {}^{u_i}K_n$ having the same label and also, they have the same distance (1) to every other label, which is a contradiction. As a consequence, we can suppose w.l.g. that ${}^{u_{i-1}}K_n$ is labeled with the set $\{1, ..., n - 2, a, b\}$ and ${}^{u_{i+1}}K_n$ is labeled with the set $\{1, ..., n - 2, c, d\}$ where $a, c \in \{n - 1, n\}$ and $b, d \in \{n + 1, n + 2\}$. Now we consider the following cases.

Case 1: $b = d$. For instance, $b = d = n + 1$. Hence the set ${}^{u_{i+2}}K_n$ (which is equal to ${}^{u_{i-2}}K_n$ since $t = 4$) contains a vertex with label $n + 2$. But, this is a contradiction, since there exist two vertices $u \in {}^{u_{i-1}}K_n$ and $v \in {}^{u_{i+1}}K_n$ having the same label and also, they have the same distance (1) to every other label.

Case 2: $b \neq d$. For instance $b = n + 1$ and $d = n + 2$. We consider the following subcases.

Case 2.1: $a = c$. For instance, $a = c = n - 1$. If there exists a vertex $w \in {}^{u_{i+2}}K_n$ with label n, then there exist two vertices $u \in {}^{u_i}K_n$ and $v \in {}^{u_{i+2}}K_n$ having the same label and also, they have the same distance (1) to every other label. If there exists a vertex $w' \in {}^{u_{i+2}}K_n$ with label $n + 1$, then there exist two vertices $u' \in {}^{u_i}K_n$ and $v' \in {}^{u_{i+1}}K_n$ having the same label and also, they have the same distance (1) to every other label. Finally, if there exists a vertex $w'' \in {}^{u_{i+2}}K_n$

with label $n+2$, then there exist two vertices $u'' \in {}^{u_{i-1}}K_n$ and $v'' \in {}^{u_i}K_n$ having the same label and also, they have the same distance (1) to every other label. Thus, for the set ${}^{u_{i+2}}K_n$ there are only $n-1$ possible labels, which is a contradiction.

Case 2.2: $a \neq c$. For instance $a = n-1$ and $c = n$. Hence, there are two vertices $u \in {}^{u_i}K_n$ and $v \in {}^{u_{i+2}}K_n$ having the same label and also, they have the same distance (one) to every other label, a contradiction.

As a consequence of the above cases, we have that $pd(C_4 \boxtimes K_n) \geq n+3$. For the upper bound, we consider a vertex partition $\Pi_4 = \{B_1, B_2, ..., B_{n+3}\}$ for $C_4 \boxtimes K_n$ of order $n+3$ given by $B_1 = \{(u_0, v_1)\}$, $B_2 = \{(u_1, v_1)\}$, $B_3 = \{(u_2, v_1)\}$, $B_4 = \{(u_3, v_1)\}$, $B_5 = V_1 \times \{v_2\}$, $B_6 = V_1 \times \{v_3\}$, ..., $B_{n+3} = V_1 \times \{v_n\}$. Since for any two different vertices $(u_i, v_j), (u_l, v_j)$ of $C_4 \boxtimes K_n$ there exist a vertex (u_k, v_1) such that $d_{C_4 \boxtimes K_n}((u_i, v_j), (u_k, v_1)) = 1 \neq 2 = d_{C_4 \boxtimes K_n}((u_l, v_j), (u_k, v_1))$ it is straightforward to observe that Π_4 is a resolving partition for $C_4 \boxtimes K_n$. Therefore, $pd(C_4 \boxtimes K_n) = n+3$.

We analyze now the case $t = 5$. Suppose that $pd(C_5 \boxtimes K_n) = n+2$ and, as in the case $t = 4$, let $\Pi = \{A_1, A_2, ..., A_{n+2}\}$ be a resolving partition for $C_5 \boxtimes K_n$. Consider a set ${}^{u_i}K_n$. From Remark 7 we have that every two different vertices of ${}^{u_i}K_n$ have different labels. We need to show that two consecutive layers ${}^{u_i}K_n$ and ${}^{u_{i+1}}K_n$ differ in exactly one label.

Suppose that in $C_5 \boxtimes K_n$ there exist two consecutive layers, say ${}^{u_i}K_n$ and ${}^{u_{i+1}}K_n$, such that their vertices have the same set of labels. Without loss of generality assume these labels are $\{1, 2, \ldots, n\}$. Now, take two vertices $v \in {}^{u_i}K_n$ and $w \in {}^{u_{i+1}}K_n$, with the same label ($v, w \in A_k$, for some k). Since Π is a resolving partition, these two vertices must be resolved by a vertex with label $x \in \{n+1, n+2\}$. Moreover, this vertex is in layer ${}^{u_{i-1}}K_n$ or ${}^{u_{i+2}}K_n$. Observe, that if both layers ${}^{u_{i-1}}K_n$ and ${}^{u_{i+2}}K_n$ contain a vertex with label x then these two vertices are not resolved in Π, therefore w.l.g. we can assume that the vertex with the label x resolving v and w is in layer ${}^{u_{i+2}}K_n$ (and the layer ${}^{u_{i-1}}K_n$ contains no vertex with the label x). Now, take two vertices $v' \in {}^{u_{i+1}}K_n$ and $w' \in {}^{u_{i+2}}K_n$, with the same label ($v, w \in A_{k'}$, for some k'). Clearly, they must be resolved by a vertex with label $y \in \{n+1, n+2\} \setminus \{x\}$. Moreover, this vertex is in layer ${}^{u_{i+3}}K_n$, since the layer ${}^{u_i}K_n$ contains no such vertex by assumption. Observe, that the layer ${}^{u_{i-1}}K_n$ contains no vertex with the label y, since two vertices with the same label from layers ${}^{u_i}K_n$ and ${}^{u_{i+4}}K_n$ would not be resolved. It follows that the layer ${}^{u_{i-1}}K_n$ contains labels $\{1, 2, \ldots, n\}$. Now take two vertices with the same label from the layers ${}^{u_{i+2}}K_n$ and ${}^{u_{i+3}}K_n$, clearly they are not resolved by any vertex. This is a contradiction with Π being a resolving set. It follows that no two consecutive layers contain the same set of labels. Moreover, if two consecutive K_n-layers have together all labels $\{1, 2, \ldots, n, n+1, n+2\}$, then because of $n \geq 3$ we have that $2n > n+2$, and, by the pigeonhole principle, there exist two vertices belonging to these two K_n-layers with the same label that have equal distance (1) to all other labels, which is a contradiction. Therefore, two consecutive layers cannot together contain all labels $\{1, 2, \ldots, n, n+1, n+2\}$ and it follows that the sets of labels of two consecutive layers differ in exactly one label.

In this sense two consecutive layers contain exactly $n+1$ labels. Therefore, one label is missing. Let L_i and L_{i+1} be the set of labels used in layers ${}^{u_i}K_n$ and ${}^{u_{i+1}}K_n$, respectively, and let $CL_i = \{1, \ldots, n+2\} \backslash (L_i \cup L_{i+1})$ be the set of the missing labels between consecutive layers ${}^{u_i}K_n$ and ${}^{u_{i+1}}K_n$. Notice that we have exactly five CL-sets. Without loss of generality suppose that $L_i = \{1, \ldots, n\}$ and $L_{i+1} = \{2, \ldots, n+1\}$. Then $CL_i = \{n+2\}$. We show that also CL_{i-1} or CL_{i+1} must be $\{n+2\}$. Suppose this is not true. Then the only possibility is $CL_{i-1} = \{n+1\}$

and $CL_{i+1} = \{1\}$. But then there exist vertices $v_1 \in {}^{u_{i-1}}K_n$ and $v_2 \in {}^{u_{i+2}}K_n$, both labeled with $n+2$. Since $n \geq 3$ there exist two vertices in layers ${}^{u_i}K_n$ and ${}^{u_{i+1}}K_n$ with the same label that have the same distance to the labels $\{1, \dots, n+1\}$ which are in the same layers, and they also have the same distance to vertices v_1 and v_2, respectively, which are labeled with $n+2$. Therefore, the set $\{A_1, A_2, ..., A_{n+2}\}$ is not a resolving partition, which is a contradiction. We have proved that either $CL_i = CL_{i-1}$ or $CL_i = CL_{i+1}$.

Next, we show that no more than two consecutive CL-sets can be the same. Suppose that three consecutive CL-sets are the same, *i.e.* $CL_{i-1} = CL_i = CL_{i+1}$. Then by the same argument as above there exist a vertex $v \in {}^{u_{i-2}}K_n = {}^{u_{i+3}}K_n$ labeled with the label $l \in CL_{i-1} = CL_i = CL_{i+1}$. Since $n \geq 3$ there exist two vertices in layers ${}^{u_i}K_n$ and ${}^{u_{i+1}}K_n$ with the same label, say label k, that have the same distance to the labels $\{1, \dots, k-1, k+1, \dots, n+2\} \backslash \{l\}$ which are in the same layers, and they also have the same distance to vertex v, which is labeled with l. Again, the set $\{A_1, A_2, ..., A_{n+2}\}$ is not a resolving partition, which is a contradiction. Hence, there are exactly two consecutive CL-sets which are the same. Therefore $CL_i = CL_{i+1} = \{a\}$, $CL_{i+2} = CL_{i+3} = \{b\}$, $b \neq a$, and $CL_{i+4} = c$, $c \neq a, b$. According to the observation above, we have a CL-set which is not equal to any of its neighboring CL-sets. Therefore, $pd(C_5 \boxtimes K_n) \geq n+3$.

On the other hand, we consider a vertex partition $\Pi_5 = \{B_1, B_2, ..., B_{n+3}\}$ for $C_5 \boxtimes K_n$ of order $n+3$ given by $B_1 = \{(u_0, v_1)\}$, $B_2 = \{(u_1, v_1)\}$, $B_3 = \{(u_2, v_1)\}$, $B_4 = \{(u_3, v_1), (u_4, v_1)\}$, $B_5 = V_1 \times \{v_2\}$, $B_6 = V_1 \times \{v_3\}$, ..., $B_{n+3} = V_1 \times \{v_n\}$. Consider two different vertices $(u_i, v_j), (u_l, v_j)$ of $C_5 \boxtimes K_n$. If $(u_i, v_j), (u_l, v_j)$ are the vertices $\{(u_3, v_1), (u_4, v_1)\}$ of B_4, then they are resolved by B_1 or B_3. On the contrary, notice that there always exists a vertex (u_k, v_1), $k \in \{0, 1, 2\}$, such that $d_{C_5 \boxtimes K_n}((u_i, v_j), (u_k, v_1)) = 1 \neq 2 = d_{C_5 \boxtimes K_n}((u_l, v_j), (u_k, v_1))$. Thus $(u_i, v_j), (u_l, v_j)$ are resolved by B_1, B_2 or B_3 and, as a consequence, Π_5 is a resolving partition for $C_5 \boxtimes K_n$. Therefore, $pd(C_5 \boxtimes K_n) = n+3$.

From now on we suppose $t \geq 6$. Let $\Pi = \{A_1, A_2, ..., A_{n+2}\}$ be the vertex partition, where

$$
\begin{aligned}
A_1 &= \{(u_0, v_1)\}, \\
A_2 &= \{(u_1, v_1), (u_2, v_1)\}, \\
A_3 &= \{(u_3, v_1), (u_4, v_1), ..., (u_{t-1}, v_1)\}, \\
A_4 &= \{(u_0, v_2), (u_1, v_2), ..., (u_{t-1}, v_2)\}, \\
A_5 &= \{(u_0, v_3), (u_1, v_3), ..., (u_{t-1}, v_3)\}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
A_{n+2} &= \{(u_0, v_n), (u_1, v_n), ..., (u_{t-1}, v_n)\}.
\end{aligned}
$$

Figure 2 shows the partition for the case of $C_6 \boxtimes K_4$.

We claim that Π is a resolving partition for $C_t \boxtimes K_n$. Let $(u_i, v_j), (u_l, v_j) \in V_1 \times V_2$ be two different vertices belonging to the same set of the partition Π. First, if $i, l \in \{0, 1\}$ and $i \neq l$, then $(u_i, v_j), (u_l, v_j)$ are resolved by A_3. On the contrary, since diameter of C_t is greater or equal than three, we have that if $d_{C_t \boxtimes K_n}((u_i, v_j), A_1) = d_{C_t \boxtimes K_n}((u_l, v_k), A_1)$, then we have that $d_{C_t \boxtimes K_n}((u_i, v_j), A_2) \neq d_{C_t \boxtimes K_n}((u_l, v_k), A_2)$. Thus, $(u_i, v_j), (u_l, v_j)$ are resolved by A_1 or A_2 and, as a consequence, Π is a resolving partition for $C_t \boxtimes K_n$. Therefore the result follows for $t \geq 6$. $\square$

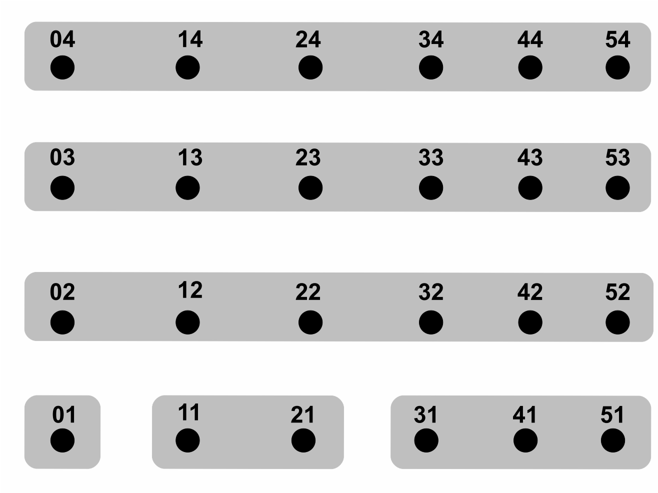

Figure 2: The partition Π of $C_6 \boxtimes K_4$. Vertex labeled by ij represents the vertex (u_i, v_j). Edges of the graph have not been drawn.

3 Cartesian product graphs

The partition dimension of the Cartesian product graphs was first studied in [17]. For instance, in that article it was obtained that for any non trivial connected graphs G and H, it follows that $pd(G\Box H) \le pd(G) + pd(H)$ and that $pd(G\Box H) \le dim(G) + pd(H)$. As we now show, these bounds can be improved.

Theorem 11. *Let G and H be two non trivial connected graphs. Then*

$$pd(G \Box H) \le pd(G) + pd(H) - 1\,.$$

Proof. Let $\Pi_1 = \{A_1, A_2, \ldots, A_{pd(G)}\}$ and $\Pi_2 = \{B_1, B_2, \ldots, B_{pd(H)}\}$ be a resolving partition of the graphs G and H, respectively. With these two partitions we make the partition $\Pi = \{A_1 \times B_1, A_2 \times B_1, \ldots, A_{pd(G)} \times B_1, V(G) \times B_2, \ldots, V(G) \times B_{pd(H)}\}$ of the graph $G \Box H$ with cardinality $pd(G) + pd(H) - 1$. Next we prove that Π is a resolving partition of $G \Box H$. We consider two different vertices $(u_1, v_1), (u_2, v_2) \in V(G \Box H)$.

Case 1: If $(u_1, v_1), (u_2, v_2) \in A_i \times B_1$ for some $i \in \{1, ..., pd(G)\}$, then we consider the following subcases.

Case 1.1: $u_1 = u_2$. Hence $v_1 \ne v_2$ and there exists a set $B_j \in \Pi_2$ such that $d_H(v_1, B_j) \ne d_H(v_2, B_j)$. Thus we have that

$$d_{G\Box H}((u_1, v_1), (V(G) \times B_j)) = d_H(v_1, B_j) \ne d_H(v_2, B_j) = d_{G\Box H}((u_2, v_2), (V(G) \times B_j)).$$

Case 1.2: $u_1 \ne u_2$. Hence there exists a set $A_l \in \Pi_1$, $l \ne i$, such that $d_G(u_1, A_l) \ne d_G(u_2, A_l)$. Thus we have that

$$d_{G\Box H}((u_1, v_1), (A_l \times B_1)) = d_G(u_1, A_l) \ne d_G(u_2, A_l) = d_{G\Box H}((u_2, v_2), (A_l \times B_1)).$$

Case 2: $(u_1, v_1), (u_2, v_2) \in A_i \times B_j$, $j \neq 1$ and $v_1 \neq v_2$. Then there exists a set B_k, $k \neq j$, such that $d_H(v_1, B_k) \neq d_H(v_2, B_k)$. If $k \neq 1$, then we have that

$$d_{G\Box H}((u_1, v_1), V(G) \times B_k) = d_H(v_1, B_k) \neq d_H(v_2, B_k) = d_{G\Box H}((u_2, v_2), V(G) \times B_k).$$

On the contrary, if $k = 1$, then

$$d_{G\Box H}((u_1, v_1), A_i \times B_k) = d_H(v_1, B_k) \neq d_H(v_2, B_k) = d_{G\Box H}((u_2, v_2), A_i \times B_k).$$

Case 3: $(u_1, v_1), (u_2, v_2) \in A_i \times B_j$, $j \neq 1$ and $v_1 = v_2$. Then there exists a set A_l, $l \neq i$, such that

$$\begin{aligned} d_{G\Box H}((u_1, v_1), A_l \times B_1) &= d_G(u_1, A_l) + d_H(v_1, B_1) \\ &= d_G(u_1, A_l) + d_H(v_2, B_1) \\ &\neq d_G(u_2, A_l) + d_H(v_2, B_1) \\ &= d_{G\Box H}((u_2, v_2), A_l \times B_1), \end{aligned}$$

and the inequality holds since Π_1 is a resolving partition of G.

Case 4: $(u_1, v_1) \in A_i \times B_j$ and $(u_2, v_2) \in A_l \times B_j$, $j \neq 1$ and $l \neq i$. Without loss of generality assume that $d(v_1, B_1) \geq d(v_2, B_1)$. Then

$$\begin{aligned} d_{G\Box H}((u_1, v_1), A_l \times B_1) &= d_G(u_1, A_l) + d_H(v_1, B_1) \\ &\geq d_G(u_1, A_l) + d_H(v_2, B_1) \\ &\geq d_G(u_2, A_l) + 1 + d_H(v_2, B_1) \\ &= d_{G\Box H}((u_2, v_2), A_l \times B_1) + 1 \end{aligned}$$

and the second inequality holds since $d(u_2, A_l) = 0$ and $d(u_1, A_l) \geq 1$. As a consequence of the above cases we have that the vertices (u_1, v_1) and (u_2, v_2) are resolved by some set of Π. Therefore, Π is a resolving partition for $G\Box H$ and the proof is complete. □

The above bound is tight. For instance, for any two paths P_m, P_n with $m, n \geq 2$, it is satisfied $pd(P_m\Box P_n) = pd(P_m) + pd(P_n) - 1 = 3$. Also, notice that as a consequence of Theorem 11 and by inequality 1, we have that $pd(G\,\Box\,H) \leq pd(G) + dim(H)$, which is the bound obtained in [17].

Corollary 12. *Let $r, t \geq 2$ be integers. Then*

- $pd(P_r\Box P_t) = 3$,
- $pd(P_r\Box C_t) \leq 4$,
- $pd(C_r\Box C_t) \leq 5$ *and*
- *for any graph H, $pd(K_r\Box H) \leq r + pd(H) - 1$.*

Even though the upper bound of Theorem 11 is tight, it is possible to improve it for several cases, as Theorem 13 shows.

Theorem 13. *Let H be a non trivial connected graph of order t and let $n \geq 3$. Then*

$$pd(K_n\,\Box\,H) \leq \min\left\{ \left\lceil \frac{n}{k} \right\rceil (pd(H) - 1) + k \,:\, 2 \leq k \leq n - 1 \right\}.$$

Proof. Let $U = \{u_0, u_1, ..., u_{n-1}\}$ be the vertex set of K_n and let $k \in \{2, ..., n-1\}$ be an integer. Hence, we have that $n = k \cdot p + q$ for some p, q being non negative integers and $q \leq k-1$. Notice that $p = \left\lfloor \frac{n}{k} \right\rfloor$. Let $\Pi_2 = \{X_1, X_2, ..., X_r\}$ be a resolving partition for H. We consider the vertex partition Π for $K_n \,\square\, H$ given by (see Figure 3 for a geometrical representation of one example):

$$\begin{aligned}
A_0^1 &= \{u_0, u_1, ..., u_{k-1}\} \times X_1, \\
A_1^1 &= \{u_k, u_{k+1}, ..., u_{2k-1}\} \times X_1, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
A_{p-1}^1 &= \{u_{(p-1)\cdot k}, ..., u_{pk-1}\} \times X_1, \\
A_p^1 &= \{u_{p\cdot k}, ..., u_n\} \times X_1, \\
A_0^2 &= \{u_0, u_1, ..., u_{k-1}\} \times X_2, \\
A_1^2 &= \{u_k, u_{k+1}, ..., u_{2k-1}\} \times X_2, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
A_{p-1}^2 &= \{u_{(p-1)\cdot k}, ..., u_{pk-1}\} \times X_2, \\
A_p^2 &= \{u_{p\cdot k}, ..., u_n\} \times X_2, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
A_0^{r-1} &= \{u_0, u_1, ..., u_{k-1}\} \times X_{r-1}, \\
A_1^{r-1} &= \{u_k, u_{k+1}, ..., u_{2k-1}\} \times X_{r-1}, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
A_{p-1}^{r-1} &= \{u_{(p-1)\cdot k}, ..., u_{pk-1}\} \times X_{r-1}, \\
A_p^{r-1} &= \{u_{p\cdot k}, ..., u_n\} \times X_{r-1},
\end{aligned}$$

and

$$\begin{aligned}
B_0 &= \{u_0, u_k, ..., u_{p\cdot k}\} \times X_r, \\
B_1 &= \{u_1, u_{k+1}, ..., u_{p\cdot k+1}\} \times X_r, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
B_{q-1} &= \{u_{q-1}, u_{k+q-1}, ..., u_{p\cdot k+q-1}\} \times X_r, \\
B_q &= \{u_q, u_{k+q}, ..., u_{(p-1)k+q}\} \times X_r, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
B_{k-1} &= \{u_{k-1}, u_{2k-1}, ..., u_{p\cdot k-1}\} \times X_r.
\end{aligned}$$

We claim that Π is a resolving partition for $K_n \square H$. We consider two different vertices $(u_i, v_j), (u_g, v_h)$ of $K_n \square H$ belonging to the same set of the partition Π. We have the following cases.

Case 1: $i = g$. Hence $j \neq h$ and there exists a set $X_l \in \Pi_2$ such that $d_H(v_j, X_l) \neq d_H(v_h, X_l)$. If $l \neq r$, then we have that

$$d_{K_n \square H}\left((u_i, v_j), A^j_{\lfloor i/k \rfloor}\right) = d_H(v_j, X_l) \neq d_H(v_h, X_l) = d_{K_n \square H}\left((u_g, v_h), A^j_{\lfloor i/k \rfloor}\right).$$

On the contrary, if $j = r$, then we have that

$$d_{K_n \square H}((u_i, v_j), B_{i \bmod k}) = d_H(v_j, X_r) \neq d_H(v_j, X_r) = d_{K_n \square H}((u_g, v_h), B_{g \bmod k}).$$

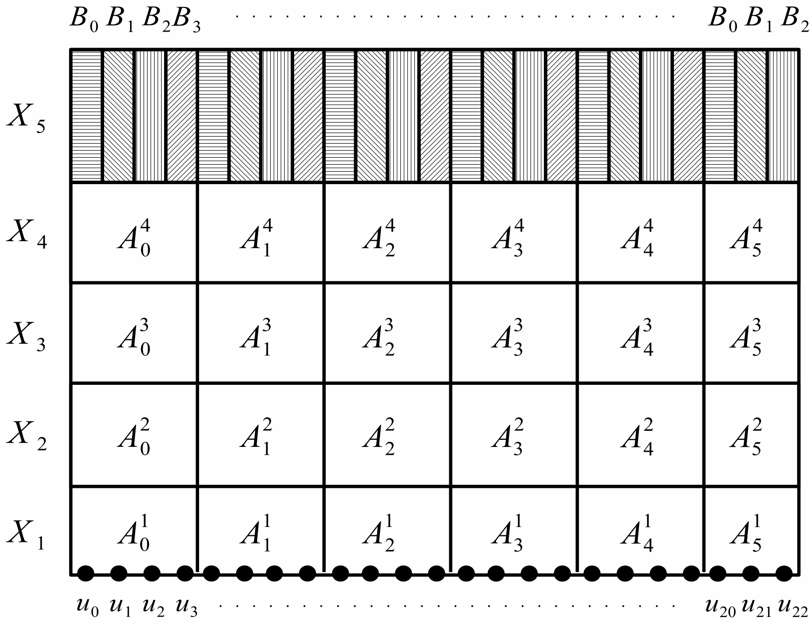

Figure 3: A resolving partition of $K_{23}\Box H$ with cardinality $\left\lceil \frac{n}{k} \right\rceil (pd(H)-1)+k = 28$ for $k=4$ and $pd(H)=5$. Notice that in this case $p=5$ and $q=3$. The sets A_i^j are labeled and the sets B_l are given by the union of rectangles with identically filled areas.

Case 2: $i \neq g$. Hence we consider the following subcases.

Case 2.1: $v_j, v_h \in X_r$. Hence for any $l \in \{1,...,r-1\}$ it follows that either

$$d_{K_n\Box H}\left((u_i,v_j), A^l_{\lfloor i/k \rfloor}\right) \neq d_{K_n\Box H}\left((u_g,v_h), A^l_{\lfloor i/k \rfloor}\right).$$

or

$$d_{K_n\Box H}\left((u_i,v_j), A^l_{\lfloor g/k \rfloor}\right) \neq d_{K_n\Box H}\left((u_g,v_h), A^l_{\lfloor g/k \rfloor}\right).$$

Case 2.2: $v_j, v_h \notin X_r$. Similarly, we have either

$$d_{K_n\Box H}((u_i,v_j), B_{i \bmod k}) \neq d_{K_n\Box H}((u_g,v_h), B_{i \bmod k}),$$

or

$$d_{K_n\Box H}((u_i,v_j), B_{g \bmod k}) \neq d_{K_n\Box H}((u_g,v_h), B_{g \bmod k}).$$

Therefore, Π is a resolving partition for $G\Box H$. Now, since the cardinality of the partition Π is given by $\left\lceil \frac{n}{k} \right\rceil (pd(H)-1)+k$ (there are $\left\lceil \frac{n}{k} \right\rceil (pd(H)-1)$ sets of type A and k sets of type B) and this is satisfied for every value of $k \in \{2,...,n-1\}$, we obtain that $pd(K_n\Box H) \leq \min\left\{\left\lceil \frac{n}{k} \right\rceil (pd(H)-1)+k \,:\, 2\leq k \leq n-1\right\}$. Therefore the proof is complete. □

Notice that for instance, if H is a path P_t, then Theorem 11 leads to $pd(K_n\Box P_t) \leq n+1$, while Theorem 13 gives the value $2\sqrt{n}$, since the minimum of the expression $\left\lceil \frac{n}{k} \right\rceil (pd(H)-1)+k$ is obtained for $k=\sqrt{n}$.

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References

[1] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, and D. R. Wood, On the metric dimension of Cartesian product of graphs, *SIAM Journal of Discrete Mathematics* **21** (2) (2007) 273–302.

[2] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, and C. Seara, On the metric dimension of some families of graphs, *Electronic Notes in Discrete Mathematics* **22** (2005) 129–133.

[3] G. Chappell, J. Gimbel, and C. Hartman, Bounds on the metric and partition dimensions of a graph, *Ars Combinatoria* **88** (2008) 349–366.

[4] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Applied Mathematics* **105** (2000) 99–113.

[5] G. Chartrand, C. Poisson, and P. Zhang, Resolvability and the upper dimension of graphs, *Computers & Mathematics with Applications* **39** (2000) 19–28.

[6] G. Chartrand, E. Salehi, and P. Zhang, The partition dimension of a graph, *Aequationes Mathematicae* (1-2) **59** (2000) 45–54.

[7] M. Fehr, S. Gosselin, and O. R. Oellermann, The partition dimension of Cayley digraphs, *Aequationes Mathematicae* **71** (2006) 1–18.

[8] F. Harary and R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria* **2** (1976) 191–195.

[9] B. L. Hulme, A. W. Shiver, and P. J. Slater, A Boolean algebraic analysis of fire protection, *Algebraic and Combinatorial Methods in Operations Research* **95** (1984) 215–227.

[10] M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *Journal of Biopharmaceutical Statistics* **3** (1993) 203–236.

[11] M. A. Johnson, Browsable structure-activity datasets, *Advances in Molecular Similarity* (R. Carbó–Dorca and P. Mezey, eds.) JAI Press Connecticut (1998) 153–170.

[12] S. Khuller, B. Raghavachari, and A. Rosenfeld, Landmarks in graphs, *Discrete Applied Mathematics* **70** (1996) 217–229.

[13] R. A. Melter and I. Tomescu, Metric bases in digital geometry, *Computer Vision Graphics and Image Processing* **25** (1984) 113–121.

[14] J. A. Rodríguez-Velázquez, I. G. Yero, and D. Kuziak, The partition dimension of corona product graphs. *Ars Combinatoria*. (2012) In press.

[15] P. J. Slater, Leaves of trees, Proc. 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, *Congressus Numerantium* **14** (1975) 549–559.

[16] I. Tomescu, Discrepancies between metric and partition dimension of a connected graph, *Discrete Mathematics* **308** (2008) 5026–5031.

[17] I. G. Yero and J. A. Rodríguez-Velázquez, A note on the partition dimension of Cartesian product graphs. *Applied Mathematics and Computation* **217** (7) (2010) 3571-3574.