On the vertex k-path cover

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Abstract

A subset S of vertices of a graph G is called a vertex k-path cover if every path of order k in G contains at least one vertex from S. Denote by $\psi_k(G)$ the minimum cardinality of a vertex k-path cover in G. In this paper an upper bound for ψ_3 in graphs with a given average degree is presented, A lower bound for ψ_k of regular graphs is also proven. For grids, i.e. the Cartesian products of two paths, we give an asymptotically tight bound for ψ_k and the exact value for ψ_3.

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1 Introduction

Let G be a graph and k be a positive integer. Then $S \subseteq V(G)$ is a *vertex k-path cover* of G if every path on k vertices in G contains a vertex from S. We denote by $\psi_k(G)$ the minimum cardinality of a vertex k-path cover in G. This graph invariant was recently introduced in [4], motivated by the problem of ensuring data integrity of communication in wireless sensor networks using the k-generalized Canvas scheme [13].

The concept of vertex k-path cover is a generalization of the vertex cover. Clearly $\psi_2(G)$ coincides with the minimum cardinality of a vertex cover in a graph G, and so

$$\psi_2(G) = |V(G)| - \alpha(G)$$

where $\alpha(G)$ stands for the independence number. In addition $\psi_3(G)$ corresponds to another previously studied concept of *dissociation number* of a graph [17], defined as follows. A subset of vertices in a graph G is called a *dissociation* set if it induces a subgraph with maximum degree at most 1. The maximum cardinality of a dissociation set in G is called the *dissociation number* of G and is denoted $diss(G)$. Clearly

$$\psi_3(G) = |V(G)| - diss(G).$$

The problem of computing $diss(G)$ has been introduced by Yannakakis [17], who also proved it to be NP-hard in the class of bipartite graphs. The dissociation number problem was also studied in [1, 2, 6, 8, 14, 12], see [14] for a survey.

Recently, Tu and Zhou [15] presented a 2-approximation for the 3-path vertex cover problem even in the weighted version of the problem. An exact algorithm for computing $\psi_3(G)$ in running time $O(1.5171^n)$ for a graph of order n was presented in [10]. The problem of computing $\psi_k(G)$ is NP-hard for each $k \geq 2$, but polynomial for instance in trees, as shown in [4]. The authors investigate upper bounds on the value of $\psi_k(G)$ and provide several estimations and exact values of $\psi_k(G)$. It is also proven that $\psi_3(G) \leq (2n+m)/6$, for every graph G with n vertices and m edges.

In this paper, we present a more general result by showing that for an arbitrary graph G with n vertices and m edges, and an integer k such that $k \leq \frac{m}{n} \leq k+1$, we have $\psi_3(G) \leq \frac{kn}{k+2} + \frac{m}{(k+1)(k+2)}$. This result is proven in Section 2, while in Section 3 we consider bounds for d-regular graphs. We show that for an arbitrary integer $k \geq 2$ and a d-regular graph G, $d \geq k-1$, we have $\psi_k(G) \geq \frac{d-k+2}{2d-k+2}|V(G)|$. Section 4 is devoted to the vertex k-path cover number of Cartesian products of graphs, with an emphasis on grids, for which we present the exact value for ψ_3.

We conclude this section with the following straightforward values of ψ_k.

Proposition 1.1. *Let $k \geq 2$ and $n \geq k$ be positive integers. Then*

- $\psi_k(P_n) = \lfloor \frac{n}{k} \rfloor$,
- $\psi_k(C_n) = \lceil \frac{n}{k} \rceil$,
- $\psi_k(K_n) = n - k + 1$.

2 Upper bound in terms of average degree

First, recall a result from [4] which is a consequence of Lovász's decomposition [11] of a graph with maximum degree Δ into subgraphs of maximum degree 1.

Lemma 2.1 ([4])**.** *Let G be a graph of maximum degree Δ. Then*

$$\psi_3(G) \leq \frac{\lceil \frac{\Delta-1}{2} \rceil}{\lceil \frac{\Delta+1}{2} \rceil} |V(G)|.$$

Lemma 2.2 ([4])**.** *Let G be a graph on n vertices and m edges. Then*

$$\psi_3(G) \leq \frac{2n+m}{6}.$$

In the following theorem we give a tight upper bound in terms of the average degree of a graph, which extends the result from Lemma 2.2.

Theorem 2.1. *Let G be a graph of order n, size m and average degree $d(G)$, and k be the smallest positive integer such that $d(G) \leq 2k+2$. Then*

$$\psi_3(G) \leq \frac{kn}{k+2} + \frac{m}{(k+1)(k+2)}$$

Proof. Proof by induction on k (the basis of induction $k = 1$ coincides with the bound from Lemma 2.2).

Assume
$$\psi_3(G) \leq \frac{(k-1)n}{k+1} + \frac{m}{k(k+1)}$$

for all graphs with average degree d, $d \leq 2k$. We aim to prove that for a graph G with average degree d, $d \leq 2k+2$ the following holds

$$\psi_3(G) \leq \frac{kn}{k+2} + \frac{m}{(k+1)(k+2)}.$$

Repeatedly remove from G a vertex of degree at least $2(k+1)$, as long as such a vertex exists. In this way we get a new graph, say G', with n' vertices and m' edges. For G' we now look at two cases.

First, if $kn' < m'$, then by Lemma 2.1

$$
\begin{aligned}
\psi_3(G') &\leq \frac{k}{k+1}n' \\
&= \frac{k(k+2)}{(k+1)(k+2)}n' \\
&= \frac{k(k+1)+k}{(k+1)(k+2)}n' \\
&\leq \frac{k}{k+2}n' + \frac{m'}{(k+1)(k+2)}
\end{aligned}
$$

In the second case, when $kn' \geq m'$, we use the induction hypothesis.

$$
\begin{aligned}
\psi_3(G') &\leq \frac{k-1}{k+1}n' + \frac{m'}{k(k+1)} \\
&= \frac{k-1}{k+1}n' + \frac{2m'}{k(k+1)(k+2)} + \frac{m'}{(k+1)(k+2)} \\
&\leq \frac{k-1}{k+1}n' + \frac{2n'}{(k+1)(k+2)} + \frac{m'}{(k+1)(k+2)} \\
&= \frac{k}{k+2}n' + \frac{m'}{(k+1)(k+2)}
\end{aligned}
$$

□

Next we prove a lower bound for $\psi_3(G)$, showing that the upper bound of Theorem 2.1 is tight.

Proposition 2.1. *For every positive rational number $\frac{a}{b} > 1$, $a, b \in \mathbb{N}$, and the smallest positive integer k, such that $\frac{a}{b} \leq 2k+2$, there exists a graph G with average degree $d(G) = \frac{a}{b}$ and*

$$\psi_3(G) \geq \frac{kn}{k+2} + \frac{m}{(k+1)(k+2)}.$$

Proof. Denote by H_n a complete graph on n vertices without edges of one perfect matching (assume n is even). Clearly $|V(H_n)| = n$ and

$$|E(H_n)| = \frac{n(n-1)}{2} - \frac{n}{2} = \frac{n(n-2)}{2}.$$

Clearly $diss(H_n) = 2$, because any arbitrary three vertices of H_n form a path of order three. Therefore $\psi_3(H_n) = n - 2$, for $n \geq 2$.
We construct the graph G as the disjoint union of

- x components H_{2k+2},
- y components H_{2k+4}.

We let $x = (2b - a + 2kb)(k+2)$ and $y = (a - 2kb)(k+1)$.

First, we verify the average degree of the graph G. There are $x(2k+2)$ vertices of degree $2k$ and $y(2k+4)$ vertices of degree $2k+2$, hence we get

$$\begin{aligned}
d(G) &= \frac{x(2k+2)(2k) + y(2k+4)(2k+2)}{x(2k+2) + y(2k+4)} \\
&= \frac{(2b-a+2kb)(k+2)(2k+2)(2k) + (a-2kb)(k+1)(2k+4)(2k+2)}{(2b-a+2kb)(k+2)(2k+2) + (a-2kb)(k+1)(2k+4)} \\
&= \frac{(2b-a+2kb)(2k) + (a-2kb)(2k+2)}{(2b-a+2kb) + (a-2kb)} \\
&= \frac{4kb - (a-2kb)(2k) + (a-2kb)(2k) + 2(a-2kb)}{2b} \\
&= \frac{a}{b}.
\end{aligned}$$

Now, let us verify the value of ψ_3 of G. Clearly,

$$\psi_3(G) = x \cdot \psi_3(H_{2k+2}) + y \cdot \psi_3(H_{2k+4}) = x(2k) + y(2k+2).$$

And this exactly matches the right-hand side of inequality of the theorem:

$$\begin{aligned}
\psi_3(G) &= \frac{kn}{k+2} + \frac{m}{(k+1)(k+2)} \\
&= \frac{k(x(2k+2) + y(2k+4))}{k+2} + \frac{x(2k)(2k+2) + y(2k+2)(2k+4)}{2(k+1)(k+2)} \\
&= \frac{kx(2k+2)}{k+2} + 2ky + \frac{x(2k) + y(2k+4)}{(k+2)} \\
&= \frac{kx(2k+2)}{k+2} + 2ky + \frac{x(2k)}{(k+2)} + 2y \\
&= 2kx + 2ky + 2y.
\end{aligned}$$

□

3 Regular graphs

In the main theorem of this section we shall use the following result of Erdős and Gallai [7].

Theorem 3.1 ([7]). *If G is a graph on n vertices that does not contain a path of order k, then it cannot have more than $\frac{n(k-2)}{2}$ edges. Moreover, the bound is achieved when the graph consists of disjoint cliques on $k-1$ vertices.*

Theorem 3.2. *Let $k \geq 2$ and $d \geq k-1$ be positive integers. Then, for any d-regular graph G, the following holds:*

$$\psi_k(G) \geq \frac{d-k+2}{2d-k+2}|V(G)|.$$

Proof. Let $S \subseteq V(G)$ be a vertex k-path cover and $T = V(G) \setminus S$. Let E_S, E_T be the set of edges with both endvertices in S and T, respectively. Let E_{ST} be the set of edges with one endvertex in S and the second vertex in T. Then obviously $|E(G)| = \frac{1}{2}d|V(G)| = |E_S| + |E_{ST}| + |E_T|$.

Since G is d-regular, $d|S| = 2|E_S| + |E_{ST}|$. Therefore $|S| \geq \frac{1}{d}|E_{ST}|$. Similarly, $|E_{ST}| + 2|E_T| = d|T|$. Since the graph induced on the set E_T does not contain a path of order k, according to Theorem 3.1, we have $|E_T| \leq \frac{|T|(k-2)}{2}$. Combining all the previous formulas, we immediately have

$$|S| \geq \frac{1}{d}|E_{ST}| = \frac{1}{d}(d|T| - 2|E_T|) \geq \frac{1}{d}(d|T| - |T|(k-2)) = \frac{d-k+2}{d}|T|.$$

Then

$$\frac{|S|+|T|}{|S|} = 1 + \frac{|T|}{|S|} \leq 1 + \frac{d}{d-k+2} = \frac{2d-k+2}{d-k+2}$$

and

$$|S| \geq \frac{d-k+2}{2d-k+2}|V(G)|.$$

□

Corollary 3.1. *If G is a cubic graph then $\psi_k(G) \geq \frac{5-k}{8-k}|V(G)|$, in particular $\psi_3(G) \geq \frac{2}{5}|V(G)|$.*

Corollary 3.2. *Let $k \geq 2$ be a fixed positive integer and Q_d the d-dimensional hypercube. Then*

$$\lim_{d\to\infty} \frac{\psi_k(Q_d)}{|V(Q_d)|} = \frac{1}{2}.$$

Proof. By Theorem 3.2, we know that $\frac{d-k+2}{2d-k+2}|V(Q_d)| \leq \psi_k(Q_d) \leq \psi_2(Q_d) \leq \frac{1}{2}|V(Q_d)|$, and the result immediately follows. □

Proposition 3.1. *Let $k \geq 2$ and $d \geq k-1$ be positive integers. There exists a d-regular graph G with*

$$\psi_k(G) \leq \frac{d-k+2}{2d-k+2}|V(G)|.$$

Proof. Given fixed values k and d, a graph G may be constructed as follows. Let A be a graph on $(k-1)(d-k+2)$ independent vertices and B a graph consisting of d compontents of K_{k-1}. Let G be a disjoint union of A and B, together with addition edges between them, which can be arbitrarily chosen to fill d-regularity of G. Clearly, A is a k-path vertex cover of G since B does not contain a path on k vertices and

$$\frac{\psi_k(G)}{|V(G)|} \leq \frac{|A|}{|A|+|B|} = \frac{(k-1)(d-k+2)}{(k-1)(d-k+2)+d(k-1)} = \frac{d-k+2}{2d-k+2}.$$

□

4 Cartesian products

Recall that the *Cartesian product* $G \,\Box\, H$ of graphs $G = (V(G), E(G))$ and $H = (V(H),\ E(H))$ has the vertex set $V(G) \times V(H)$, and vertices $(u, v), (x, y) \in V(G \,\Box\, H)$ are adjacent whenever $u = x$ and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$. By p_G and p_H we denote the natural projections to the factors G and H, respectively. For a fixed vertex $u \in V(G)$, the *H-layer* ${}^{u}H$ is the subgraph of $G\Box H$ induced by the set of vertices $\{(u, v),\ v \in V(H)\}$; analogously, for a fixed vertex $v \in V(H)$, the *G-layer* G^v is the subgraph of $G\Box H$ induced by the set of vertices $\{(u, v),\ u \in V(G)\}$.

Note that each H-layer is isomorphic to H, and is an induced subgraph of $G\Box H$. Hence any vertex k-path cover S of $G\Box H$ must contain at least $\psi_k(H)$ vertices in every H-layer. Much more can be said if we restrict the structure of one of the factors. In the main result of this section we focus on the ψ_3 of grids (Cartesian products of paths), for which exact formulas are obtained.

Theorem 4.1. *(i)* $\psi_3(P_{2n+1} \,\Box\, P_{2k}) = 2nk + \lfloor \frac{2k}{3} \rfloor$, *where* $n, k \geq 1$,

(ii) $\psi_3(P_{2n} \,\Box\, P_{2k}) = 2nk$, *where* $n, k \geq 1$,

(iii) $\psi_3(P_{2n+1} \,\Box\, P_{2k+1}) = n(2k+1) + \lfloor \frac{2k+1}{3} \rfloor$, *where* $1 \leq n \leq k$.

Proof. (i) Let us label the vertices of P_{2n+1} consecutively by $u_1, u_2, \ldots, u_{2n+1}$ and the vertices of P_{2k} by $v_1, v_2, \ldots, v_{2k}$. With this labeling, the vertex $(u_i, v_j) \in V(P_{2n+1} \,\Box\, P_{2k})$ is in i^{th} layer ${}^{u_i}P_{2k}$ and in j^{th} layer $P_{2n+1}^{v_j}$.

Let us first prove that $\psi_3(P_{2n+1} \,\Box\, P_{2k}) \leq 2nk + \lfloor \frac{2k}{3} \rfloor$. We construct a vertex 3-path cover set S in the following way:

$$S = \{(u_{2i+1}, v_{3j+3}), (u_{2i}, v_{3j+1}), (u_{2i}, v_{3j+2}) | \text{ for applicable indices } i \text{ and } j\}.$$

It is not difficult to verify that S is a vertex 3-path cover set (Fig. 1) and gives the desired upper bond.

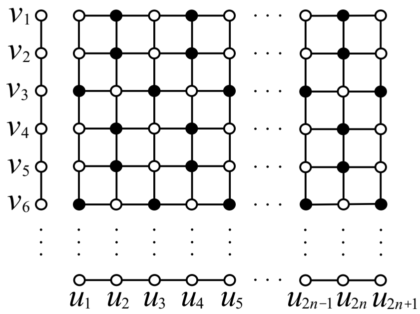

Figure 1: A vertex 3-path cover of $P_{2n+1} \,\Box\, P_{2k}$

Now let us prove that $\psi_3(P_{2n+1} \,\Box\, P_{2k}) \geq 2nk + \lfloor \frac{2k}{3} \rfloor$. Let S be the optimal vertex 3-path cover. Consider the layer ${}^{u_i}P_{2k}$, for some $i \in \{1, 2, \ldots, 2n\}$.

Suppose $(u_i, v_j) \notin S$, for some $j \in \{1, 2, \ldots, 2k-1\}$, and all its neighbors in ${}^{u_i}P_{2k}$ are in S. Then in the layer ${}^{u_{i+1}}P_{2k}$ either the vertex (u_{i+1}, v_j) is in S (left-hand side of Fig. 2) or all its neighbors in this layer are in S (right-hand side of Fig. 2).

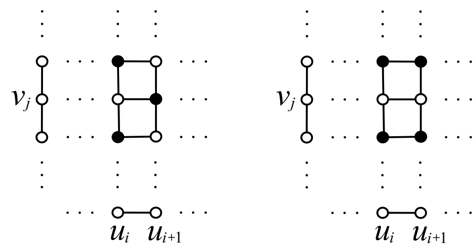

Figure 2: Vertex (u_i, v_j) is not in S

Suppose (u_i, v_j) and (u_i, v_{j+1}), for some $j \in \{1, 2, \ldots, 2k-1\}$, are not in S. Then the vertices (u_{i+1}, v_j) and (u_{i+1}, v_{j+1}) must be in S, otherwise S is not a vertex 3-path cover (see Fig. 3).

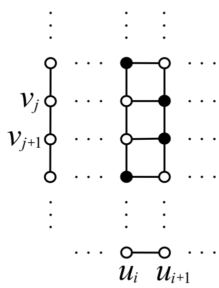

Figure 3: Vertices (u_i, v_j) and (u_i, v_{j+1}) are not in S

Let A be the set of all vertices in layer ${}^{u_i}P_{2k}$ that are not in S, and B the set of all vertices in layer ${}^{u_{i+1}}P_{2k}$ that are in S. We shall prove that

$|A| \leq |B|$ by finding a one to one function f from A to B. Let (u_i, v_j) be a vertex in A.

Suppose that the vertex (u_i, v_{2k}) is not in A. By Fig. 2 and Fig. 3 we set $f((u_i, v_j)) = (u_{i+1}, v_j)$ if (u_{i+1}, v_j) is in S, otherwise we set $f((u_i, v_j)) = (u_{i+1}, v_{j+1})$. This is obviously a one to one function.

We take a different approach if the vertex (u_i, v_{2k}) is in A. If also the vertex (u_i, v_{2k-1}) is in A, we can use the same argument as above since according to Fig. 3, we can set $f((u_i, v_{2k})) = (u_{i+1}, v_{2k})$. If the vertex $(u_i, v_{2k-1}) \notin A$ and $(u_{i+1}, v_{2k}) \in B$, then we can set $f((u_i, v_{2k})) = (u_{i+1}, v_{2k})$, otherwise we can set $f((u_i, v_{2k})) = (u_{i+1}, v_{2k-1})$. It may occur that the vertex (u_{i+1}, v_{2k-1}) is already an image of the vertex (u_i, v_{2k-2}). If this situation occurs we can set either $f((u_i, v_{2k-2})) = (u_{i+1}, v_{2k-2})$ or $f((u_i, v_{2k-2})) = (u_{i+1}, v_{2k-3})$ since at least one of the vertices (u_{i+1}, v_{2k-2}) or (u_{i+1}, v_{2k-3}) is in B. Note that if vertex (u_i, v_{2k-2}) exists in the grid then so must vertex (u_{i+1}, v_{2k-3}) since every P_{2k}-layer has even number of vertices. It can occur that the vertex (u_{i+1}, v_{2k-3}) is also an image of a vertex in layer ${}^{u_i}P_{2k}$. In this case we can repeat the above procedure.

We can always find a one to one function from A to B and hence $|A| \leq |B|$. Therefore two consecutive layers, ${}^{u_i}P_{2k}$ and ${}^{u_{i+1}}P_{2k}$, must have at least $2k$ vertices in S. There are n such paired layers in $P_{2n+1} \,\square\, P_{2k}$. Together with the additional layer ${}^{u_{2n+1}}P_{2k}$, which is isomorphic to the path P_{2k}, there must be at least $2nk + \lfloor \frac{2k}{3} \rfloor$ vertices in S.

(ii) We follow the same line of thought as in *(i)*, with the exception that in this case there is no additional layer.

(iii) Note that in the proof of *(i)*, two consecutive layers, ${}^{u_i}P_{2k}$ and ${}^{u_{i+1}}P_{2k}$, have at least $2k$ vertices in S. In this case the layer ${}^{u_i}P_{2k+1}$ has odd number of vertices and it can occur that two consecutive layers do not have at least $2k + 1$ vertices in S. Nevertheless, two consecutive layers, ${}^{u_i}P_{2k+1}$ and ${}^{u_{i+1}}P_{2k+1}$, must still have at least $2k$ vertices in S. Moreover if they have exactly $2k$ vertices in S, then for layers ${}^{u_i}P_{2k+1}$ and ${}^{u_{i+1}}P_{2k+1}$ the vertices (u_i, v_{2j}) and (u_{i+1}, v_{2j}), $j \in \{1, \ldots, k\}$, must all be in S (see the structure of sets A and B in *(i)*).

Having made the basic observation let us proceed with the proof. For the upper bound the same construction as in *(i)* suffices. Now we want to prove that $\psi_3(P_{2n+1} \,\square\, P_{2k+1}) \geq n(2k+1) + \lfloor \frac{2k+1}{3} \rfloor$. Assume that $0 \leq n \leq k$ and S is the optimal vertex 3-path cover of the graph $P_{2n+1} \,\square\, P_{2k+1}$. We proceed with induction on n. For $n = 0$, the graph $P_1 \,\square\, P_{2k+1}$ is isomorphic to the path P_{2k+1}, and hence $\psi_3(P_{2k+1}) = \lfloor \frac{2k+1}{3} \rfloor$. Now let $n \geq 1$. In the induction step we consider two cases.

Suppose that the last two layers, ${}^{u_{2n}}P_{2k+1}$ and ${}^{u_{2n+1}}P_{2k+1}$, contribute at least $2k + 1$ vertices to S. Assume that $\psi_3(P_{2n-1} \,\square\, P_{2k+1}) = (n-1)(2k+1) + \lfloor \frac{2k+1}{3} \rfloor$ and $\psi_3(P_{2n+1} \,\square\, P_{2k+1}) < n(2k+1) + \lfloor \frac{2k+1}{3} \rfloor$. If we remove layers ${}^{u_{2n}}P_{2k+1}$ and ${}^{u_{2n+1}}P_{2k+1}$, we remove at least $2k + 1$ vertices from S

an therefore $\psi_3(P_{2n-1} \,\square\, P_{2k+1}) < n(2k+1) + \lfloor \frac{2k+1}{3} \rfloor - 2k - 1$. By induction assumption it follows that

$$\begin{aligned}
(n-1)(2k+1) + \left\lfloor \frac{2k+1}{3} \right\rfloor &< n(2k+1) + \left\lfloor \frac{2k+1}{3} \right\rfloor - 2k - 1 \\
2nk + n - 2k - 1 &< 2nk + n - 2k - 1 \\
0 &< 0\,,
\end{aligned}$$

which leads to a contradiction.

For the second case assume that the last two layers, ${}^{u_{2n}}P_{2k+1}$ and ${}^{u_{2n+1}}P_{2k+1}$, contribute exactly $2k$ vertices to S. According to the observation above, layer ${}^{u_{2n-1}}P_{2k+1}$ must contribute at least $k+1$ vertices to S, otherwise there exists an uncovered path P_3 in the last three layers. If we remove layers ${}^{u_{2n+1}}P_{2k+1}$, ${}^{u_{2n}}P_{2k+1}$ and ${}^{u_{2n-1}}P_{2k+1}$ (this is possible since $n \geq 1$), we remove at least $3k+1$ vertices from S. Suppose again that $\psi_3(P_{2n+1} \,\square\, P_{2k+1}) < n(2k+1) + \lfloor \frac{2k+1}{3} \rfloor$. Then $\psi_3(P_{2n-2} \,\square\, P_{2k+1}) < n(2k+1) + \lfloor \frac{2k+1}{3} \rfloor - 3k - 1$. But according to *(i)* we know that $\psi_3(P_{2n-2} \,\square\, P_{2k+1}) = 2k(n-1) + \lfloor \frac{2n-2}{3} \rfloor$ and hence

$$\begin{aligned}
2k(n-1) + \left\lfloor \frac{2n-2}{3} \right\rfloor &< n(2k+1) + \left\lfloor \frac{2k+1}{3} \right\rfloor - 3k - 1 \\
2nk - 2k + \left\lfloor \frac{2n-2}{3} \right\rfloor &< 2nk + n + \left\lfloor \frac{2k+1}{3} \right\rfloor - 3k - 1 \\
\left\lfloor \frac{2n-2}{3} \right\rfloor &< n + \left\lfloor \frac{2k+1}{3} \right\rfloor - k - 1 \\
\left\lfloor \frac{2n-2}{3} \right\rfloor - n &< \left\lfloor \frac{2k+1}{3} \right\rfloor - k - 1 \\
\left\lfloor \frac{-n-2}{3} \right\rfloor &< \left\lfloor \frac{-k-2}{3} \right\rfloor \\
k &< n
\end{aligned}$$

which again is a contradiction since $n \leq k$.

$\square$

As seen in Theorem 4.1, it is already hard to determine the exact value of ψ_3 for grids. Therefore it would be nice to have at least some lower or upper bounds for ψ_k, $k \geq 4$.

Lemma 4.1. *For each $k \geq 4$, $\psi_k(P_{2\lceil\sqrt{k}\rceil} \,\square\, P_{3\lceil\sqrt{k}\rceil}) \geq \left\lceil \sqrt{k} \right\rceil$.*

Proof. Set $G := P_{2\lceil\sqrt{k}\rceil} \,\square\, P_{3\lceil\sqrt{k}\rceil}$. Assume to the contrary that S is a k-path vertex cover of the graph G, with $|S| \leq \left\lceil \sqrt{k} \right\rceil - 1$. Then at least $\left\lceil \sqrt{k} \right\rceil$ of all $P_{3\lceil\sqrt{k}\rceil}$-layers of G do not contain any vertex of S.

Since $|S| \leq \left\lceil \sqrt{k} \right\rceil - 1$, there exists a vertex $v_j \notin S$ in the layer ${}^{u_i}P_{3\lceil \sqrt{k} \rceil}$, where $1 \leq j \leq \left\lceil \sqrt{k} \right\rceil$, such that its neighbour in the layer ${}^{u_{i+1}}P_{3\lceil \sqrt{k} \rceil}$ is also not in S. So, two consecutive $P_{3\lceil \sqrt{k} \rceil}$-layers, say ${}^{u_i}P_{3\lceil \sqrt{k} \rceil}$ and ${}^{u_{i+1}}P_{3\lceil \sqrt{k} \rceil}$, can be connected by an edge with both end-vertices not in S. Similarly, a vertex $v_l \notin S$, where $2\left\lceil \sqrt{k} \right\rceil + 1 \leq l \leq 3\left\lceil \sqrt{k} \right\rceil$, exists in ${}^{u_i}P_{3\lceil \sqrt{k} \rceil}$ and its neighbour in ${}^{u_{i+1}}P_{3\lceil \sqrt{k} \rceil}$ is also not in S.

Now, using the $P_{3\lceil \sqrt{k} \rceil}$-layers not containing any vertices of S and moving from/to the other $P_{3\lceil \sqrt{k} \rceil}$-layers on uncovered vertices only, one can easily construct a path on at least $\left\lceil \sqrt{k} \right\rceil \cdot \left\lceil \sqrt{k} \right\rceil$ vertices. Since $\left\lceil \sqrt{k} \right\rceil \cdot \left\lceil \sqrt{k} \right\rceil \geq \sqrt{k} \cdot \sqrt{k} = k$, we have a path of order at least k with no vertices in S. This is a contradiction to the assumption that S is a k-path vertex cover. □

First we present a lower bound with the help of Lemma 4.1.

Proposition 4.1. *For $k \geq 4$, $n \geq 2\left\lceil \sqrt{k} \right\rceil, m \geq 3\left\lceil \sqrt{k} \right\rceil$, the following holds*

$$\frac{nm}{24\left\lceil \sqrt{k} \right\rceil} \leq \psi_k(P_n \,\square\, P_m).$$

Proof. We partition the whole graph $P_n \,\square\, P_m$ into r disjoint subgraphs isomorphic to $P_{2\lceil \sqrt{k} \rceil} \,\square\, P_{3\lceil \sqrt{k} \rceil}$ such that $r\left(2\left\lceil \sqrt{k} \right\rceil\right)\left(3\left\lceil \sqrt{k} \right\rceil\right) \geq \frac{1}{4}nm$. By Lemma 4.1 a k-path vertex cover must have at least $\left\lceil \sqrt{k} \right\rceil$ vertices in each subgraph isomorphic to $P_{2\lceil \sqrt{k} \rceil} \,\square\, P_{3\lceil \sqrt{k} \rceil}$ in G, hence:

$$\psi_k(P_n \,\square\, P_m) \geq r\left\lceil \sqrt{k} \right\rceil \geq \frac{nm}{24\left\lceil \sqrt{k} \right\rceil}.$$

□

We conclude the paper with the following upper bound.

Proposition 4.2. *For $k \geq 4$ the following holds*

$$\psi_k(P_n \,\square\, P_m) \leq \frac{2nm}{\left\lfloor \sqrt{k} \right\rfloor} - \frac{nm}{k}.$$

Proof. We will construct a k-path vertex cover with at most $\frac{2nm}{\lfloor \sqrt{k} \rfloor} - \frac{nm}{k}$ vertices. Let $S_1 = \left\{(i,j) \in P_n \,\square\, P_m \mid i \equiv 0 \left(\mathrm{mod} \left\lfloor \sqrt{k} \right\rfloor\right)\right\}$ and similarly $S_2 = \left\{(i,j) \in P_n \,\square\, P_m \mid j \equiv 0 \left(\mathrm{mod} \left\lfloor \sqrt{k} \right\rfloor\right)\right\}$. It is easy to see that $S = (S_1 \cup S_2) \backslash (S_1 \cap S_2)$ is a k-path vertex cover, since the largest subgraph of $P_n \,\square\, P_m$ with all vertices uncovered is isomorphic to $P_{\lfloor \sqrt{k} \rfloor - 1} \,\square\, P_{\lfloor \sqrt{k} \rfloor - 1}$.

In a P_n-layer we cover every $\left\lfloor \sqrt{k} \right\rfloor$-th vertex. Since there are m layers, the size of S_1 is at most $|S_1| \leq \frac{nm}{\lfloor \sqrt{k} \rfloor}$. Similarly, $|S_2| \leq \frac{nm}{\lfloor \sqrt{k} \rfloor}$. The vertices $(i,j) \in S_1 \cap S_2$ can be left uncovered, because all the vertices $(i \pm 1, j)$ and $(i, j \pm 1)$ are in S. Since $|S_1 \cap S_2| \leq \frac{nm}{k}$, the assertion follows. $\square$

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