The Security Number of Lexicographic Products

Tanja Gologranc
University of Maribor
Faculty of Natural Sciences and Mathematics
Koroška cesta 160, 2000 Maribor, Slovenia
tanja.gologranc1@um.si

Marko Jakovac*
University of Maribor
Faculty of Natural Sciences and Mathematics
Koroška cesta 160, 2000 Maribor, Slovenia
marko.jakovac@um.si

Iztok Peterin
University of Maribor
Faculty of Electrical Engineering and Computer Science
Smetanova ulica 17, 2000 Maribor, Slovenia
iztok.peterin@um.si

March 2, 2017

Abstract

A subset S of vertices of a graph G is a secure set if $|N[X] \cap S| \geq |N[X] - S|$ holds for any subset X of S, where $N[X]$ denotes the closed neighborhood of X. The minimum cardinality $s(G)$ of a secure set in G is called the security number of G. We investigate the security number of lexicographic product graphs by defining a new concept of tightly-securable graphs. In particular we derive several exact results for different families of graphs which yield some general results.

Key words: Secure set, security number, lexicographic product of graphs.

AMS subject classification (2010): 05C69, 05C76.

1 Introduction

Secure sets and security number in graphs were introduced by Brigham *et al.* [1] while attempting to generalize the well-known concept of defensive alliances in graphs [2], which only

*Corresponding author

defend a single vertex at a given time. In general models, a more efficient defensive alliance should be able to defend any attack on the entire alliance or any part of it. After preliminary work [1] on secure sets in graphs, relatively few articles have been published regarding this topic. Some general results on security number are presented in [3, 4]. In particular in [4] it was shown that Knesser graphs $K(m,2)$ have their security number greater than half of its vertices whenever $m \geq 6$ and the number of its vertices $\binom{m}{2}$ is even. Studies of security in Cartesian product graphs were initiated already in [1], where several upper bounds were determined for grid-like graphs. Afterwards the studies continued in [7] where exact formulae and some other bounds on the security number of grid-like graphs were established. Strong product graphs were considered in [5], where the security number of grids, cylinders, and toruses was derived. Since there are no results for arbitrary graph products any step in this direction would be a nice improvement. A variation of secure sets and security number called global secure sets and global security number, respectively, were treated in [8, 9, 10] with an additional condition that the secure set must also dominate G. Global secure sets on grid-like graphs were studied in [11, 12].

The paper is structured as follows. In the remainder of this section the terminology of secure sets and lexicographic product of graphs is given. In the second section the lexicographic product with the second factor being a complete graph is considered and some exact results are determined. The third section deals with the lexicographic product with the first factor being a path or a cycle. The last section covers some general results.

In this paper $G = (V, E)$ denotes a simple graph of minimum degree δ and maximum degree Δ. For a nonempty subset $W \subseteq V$ and any vertex $v \in V$, $N_W(v)$ is the set of all vertices from W that are adjacent to v, i.e. $N_W(v) = \{u \in W \mid uv \in E(G)\}$ and $\delta_W(v) = |N_W(v)|$ denotes the degree of v in W. If $W = V$, then we use the notation $N(v)$ and call it the *open neighborhood* of v, and $\delta(v)$ which is the *degree* of v. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set W is $N(W) = \bigcup_{v \in W} N(v)$ and the *closed neighborhood* of W is $N[W] = N(W) \cup W$. The subgraph induced by a set W is denoted by $\langle W \rangle$, and the complement of W is denoted by $\overline{W}$.

The definition of a secure set is based on the following rules. Consider a set of vertices S. A vertex $y \in N[S] - S$ can attack only one neighbor in S (it does not matter if y is adjacent to several vertices in S). On the other hand, a vertex $x \in S$ can defend only one vertex in $N[x] \cap S$. We now present a formal definition of secure sets according to [1].

- For any $S = \{v_1, \ldots, v_r\} \subseteq V$, an *attack* on S is formed by any r mutually disjoint sets $A = \{A_1, \ldots, A_r\}$, for which $A_i \subseteq N_{\overline{S}}(v_i)$, $1 \leq i \leq r$.
- A *defense* of S is formed by any r mutually disjoint sets $D = \{D_1, \ldots, D_r\}$ for which $D_i \subseteq N_S[v_i]$, $1 \leq i \leq r$.
- An attack A is *defendable* if there exists a defense D such that $|D_i| \geq |A_i|$ for $1 \leq i \leq r$.
- A set S is *secure* if and only if every attack on S is defendable.

The minimum cardinality of a secure set in a graph G is the *security number* and is denoted by $s(G)$. A secure set S of cardinality $s(G)$ is called an $s(G)$-set. It is easy to see that any $s(G)$-set induces a connected subgraph. Throughout the article we use the following characterization of secure sets and the condition in the characterization will be called the *security condition*.

Theorem 1.1 [1] *A set S is a secure set in a graph G if and only if for every $X \subseteq S$,*

$$|N[X] \cap S| \geq |N[X] - S|.$$

The *lexicographic product* $G \circ H$ (also sometimes denoted by $G[H]$ and called composition) of graphs G and H is a graph with $V(G \circ H) = V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \circ H$ whenever $gg' \in E(G)$ or ($g = g'$ and $hh' \in E(H)$). For a fixed $h \in V(H)$ we call $G^h = \{(g, h) \in V(G \circ H) \,|\, g \in V(G)\}$ a *G-layer through h* in $G \circ H$. An *H-layer through g*, denoted ${}^gH$, for a fixed $g \in V(G)$ is defined symmetrically. Notice that the subgraph induced by G^h or ${}^gH$ is isomorphic to G or H, respectively. As usual we define projections $p_G : V(G \circ H) \to V(G)$ by $p_G : (g, h) \mapsto g$ and similarly $p_H : V(G \circ H) \to V(H)$ by $p_G : (g, h) \mapsto h$. The lexicographic product is clearly not commutative, while it is associative [6].

2 The second factor is a complete graph

We start by analyzing the security number of a graph $G \circ H$ where H is a complete graph. First we determine the following result.

Proposition 2.1 *If G is an arbitrary graph and $n \geq 1$, then*

$$s(G \circ K_n) \leq n \cdot s(G).$$

Proof. Let $S_1 = \{u_1, \ldots, u_{s(G)}\}$ be vertices that form a minimum secure set in graph G and set $S_2 = S_1 \times V(H)$. Let X be an arbitrary subset of S_2 and denote $Y = p_G(X)$. Since S_1 is a secure set in G we have $|N[Y] \cap S_1| \geq |N[Y] - S_1|$ and it follows that

$$|N[X] \cap S_2| = |N[Y] \cap S_1| \cdot n \geq |N[Y] - S_1| \cdot n = |N[X] - S_2|.$$

We have proved that S_2 is a secure set in graph $G \circ K_n$ and its size is exactly $|S_2| = n \cdot s(G)$. Hence, $s(G \circ K_n) \leq n \cdot s(G)$. $\square$

Now, we try to find sufficient conditions for which the bound in Proposition 2.1 is attained. First, note that $s(G \circ K_1) = s(G)$ since $G \circ K_1 \cong G$. For $n \geq 2$ we need the following definition.

Definition 2.2 *Let S be a secure set of a graph G. A subset $X \subseteq S$ is* tight *if*

$$|N[X] \cap S| = |N[X] - S|.$$

Theorem 2.3 *Let $n \geq 2$. If all minimal (with respect to inclusion) secure sets of a graph G are tight, then*

$$s(G \circ K_n) = n \cdot s(G).$$

Proof. By Proposition 2.1 we have $s(G \circ K_n) \leq n \cdot s(G)$ for any graph G. Now, suppose that S is a minimum secure set in $G \circ K_n$ where $|S| < n \cdot s(G)$. Denote with $S_1 = p_G(S)$. We analyze two cases.

Case 1: $|S_1| < s(G)$.
Obviously, S_1 is not a secure set in G, therefore there exists $Y = \{u_1, \ldots, u_k\} \subseteq S_1$ with the property $|N[Y] \cap S_1| < |N[Y] - S_1|$. For each $i \in \{1, \ldots, k\}$ there exists a $v_{j_i} \in V(K_n)$, such that $(u_i, v_{j_i}) \in S$. For $X = \{(u_1, v_{j_1}), \ldots, (u_k, v_{j_k})\} \subseteq S$ we have

$$|N[X] \cap S| \leq |N[Y] \cap S_1| \cdot n < |N[Y] - S_1| \cdot n \leq |N[X] - S|.$$

Therefore, S is not a secure set in $G \circ K_n$.

Case 2: $|S_1| \geq s(G)$.
Without loss of generality, we may assume that S_1 is a secure set in G. Otherwise, we could proceed with the same argument as in the proof of Case 1, and hence, S would not be a secure set. Let S_2 be a minimal secure set of a graph G with respect to S_1. Thus $S_2 \subseteq S_1$ and $|S_2| \geq s(G)$. By assumption S_2 is tight and we have $|S_2| = |N[S_2] \cap S_2| = |N[S_2] - S_2|$. For $X = (S_2 \times V(K_n)) \cap S$ we have

$$|N[X] \cap S| + |N[X] - S| = n \cdot (|S_2| + |N[S_2] - S_2|) = 2n|S_2|.$$

Since $|N[X] \cap S| \leq |S| < n \cdot s(G) \leq n \cdot |S_2|$, it follows that $|N[X] - S| > n \cdot |S_2|$, and hence $|N[X] \cap S| < |N[X] - S|$, which is a contradiction with the fact that S is a secure set of $G \circ K_n$.

Both cases lead to a contradiction, therefore, the assumption $|S| < n \cdot s(G)$ is not true and Proposition 2.1 implies the result. $\square$

We strongly believe that the assumption of S being "minimal" in Theorem 2.3 can be replaced with "minimum". One might also think that the assumption of S being minimal (or minimum) is not only sufficient but also necessary, but this is not the case, since the size of the smallest secure set in the lexicographic product $G \circ K_n$ does not necessary depend on the smallest secure set of G being tight, but rather on its structure. See for example the graph G in Figure 1. Its only minimum secure set (which is labeled with black color) is of size 3, i.e. $s(G) = 3$, and it is not tight. Still, it is easy to see that the security number of the lexicographic product $G \circ K_2$ equals $2 \cdot s(G) = 6$, and hence, no vertex can be removed from the secure set presented in the proof of Proposition 2.1 to make it smaller. The reason for that is that the minimum secure set of graph G is completely covered with its tight subsets. Therefore, it makes sense to define a new concept.

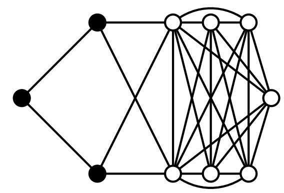

Figure 1: A graph G with minimum secure set S colored black

Definition 2.4 *Let S be any minimum secure set of a graph G and $X_1, \dots, X_n$ all of its tight subsets. If*

$$S \subseteq \bigcup_{i=1}^{n} N[X_i]$$

for any minimum secure set of G, then we say that G is tightly-securable.

All graphs with a leaf are clearly tightly-securable graphs as every leaf represents a minimum secure set which is clearly tight. Also complete graphs with even number of vertices and cycles are tightly-securable, while complete graphs with odd number of vertices are not tightly-securable. It turns out that the Definition 2.4 is necessary for the equality $s(G \circ K_n) = n \cdot s(G)$, $n \geq 2$.

Theorem 2.5 *Let $n \geq 2$. If G is not a tightly-securable graph, then*

$$s(G \circ K_n) < n \cdot s(G).$$

Proof. Let $V(G) = \{u_1, \dots, u_{|V(G)|}\}$ and $V(K_n) = \{v_1, \dots, v_n\}$. Without loss of generality assume that $S_1 = \{u_1, \dots, u_{s(G)}\}$ is a minimum secure set of G and $X_1, \dots, X_k$ all of its tight subsets. Since G is not a tightly-securable graph, we have

$$Y = \left(\bigcup_{i=1}^{k} N[X_i] \right) \cap S_1 \subsetneq S_1,$$

and there exist a vertex $x \in S_1 - Y$.

By Proposition 2.1 we know that $S_2 = S_1 \times V(K_n)$ is a secure set of a graph $G \circ K_n$. Now, let $S_3 = S_2 - \{(x, v_1)\}$. We will prove that S_3 is a secure set of $G \circ K_n$ by analyzing two cases. Let X be an arbitrary subset of S_3.

Case 1: $x \notin N[p_G(X)]$.

Then (x, v_1) is not adjacent to any vertex of X, i.e. $(x, v_1) \notin N[X]$. We know that S_2 is a secure set in $G \circ K_n$, and therefore

$$|N[X] \cap S_3| = |N[X] \cap S_2| \geq |N[X] - S_2| = |N[X] - S_3|.$$

Hence, the security condition is fulfilled for X.
Case 2: $x \in N[p_G(X)]$.
Since G is not tightly-securable and $x \in N\,[p_G(X)]$, $p_G(X)$ is not tight. Therefore

$$|N[p_G(X)] \cap S_1| \geq |N[p_G(X)] - S_1| + 1.$$

Also, $(x, v_i) \in N[X]$ for all $i \in \{1, \dots, n\}$. Note that $(x, v_1) \notin S_3$ and $(x, v_j) \in S_3$ for all $j \in \{2, \dots, n\}$. Therefore

$$\begin{aligned}
|N[X] \cap S_3| &= n \cdot |N[p_G(X)] \cap S_1| - 1 \\
&\geq n \cdot (|N[p_G(X)] - S_1| + 1) - 1 \\
&= n \cdot |N[p_G(X)] - S_1| + n - 1 \\
&= |N[X] - S_3| - 1 + n - 1 \\
&= |N[X] - S_3| + n - 2 \\
&\geq |N[X] - S_3|.
\end{aligned}$$

Again the security condition is fulfilled for X and therefore S_3 is a secure set in $G \circ K_n$ and we are done. $\square$

Corollary 2.6 *Let $n \geq 2$. If G is not a tightly-securable graph, then*

$$s(G \circ K_n) \leq n \cdot s(G) - \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. We use the same idea and notation as in the proof of Theorem 2.5, only let $S_3 = S_2 - \{(x, v_1), \dots, (x, v_{\lfloor \frac{n}{2} \rfloor})\}$. The proof of Case 1 stays the same, and the proof of Case 2 is as follows. Note that $(x, v_1), \dots, (x, v_{\lfloor \frac{n}{2} \rfloor}) \notin S_3$ and $(x, v_{\lceil \frac{n+1}{2} \rceil}), \dots, (x, v_n) \in S_3$. Therefore

$$\begin{aligned}
|N[X] \cap S_3| &= n \cdot |N[p_G(X)] \cap S_1| - \left\lfloor \frac{n}{2} \right\rfloor \\
&\geq n \cdot (|N[p_G(X)] - S_1| + 1) - \left\lfloor \frac{n}{2} \right\rfloor \\
&= n \cdot |N[p_G(X)] - S_1| + n - \left\lfloor \frac{n}{2} \right\rfloor \\
&= |N[X] - S_3| - \left\lfloor \frac{n}{2} \right\rfloor + n - \left\lfloor \frac{n}{2} \right\rfloor \\
&= |N[X] - S_3| + n - 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor \\
&\geq |N[X] - S_3|.
\end{aligned}$$

Hence, S_3 is a secure set in $G \circ K_n$. $\square$

Note, that if all minimum secure sets S of a graph G are tight, then G is also a tightly-securable graph,

$$S \subseteq \bigcup_{i=1}^{n} N[X_i],$$

since one of its tight subsets $X_1, \ldots, X_n$ is also S. This leads us to believe that the assumption of G being a tightly-securable graph is not only a necessary but also a sufficient condition for the lexicographic product $G \circ K_n$ to have its security number $s(G \circ K_n) = n \cdot s(G)$. Hence, the following conjecture.

Conjecture 2.7 *Let $n \geq 2$. A graph G is tightly-securable if and only if*

$$s(G \circ K_n) = n \cdot s(G).$$

3 The first factor is a cycle or a path

We start this section with a complete anwser for $s(C_n \circ H)$. A generalization of this result will follow in the last section. First a lemma.

Lemma 3.1 *Let $n \geq 3$. If S is a secure set in $C_n \circ H$, then $|p_{C_n}(S)| \geq 2$.*

Proof. Suppose that $|p_{C_n}(S)| = 1$ for a secure set S in $C_n \circ H$ and let (u, v) be an arbitrary vertex from S. Since $|p_{C_n}(S)| = 1$, it follows that

$$|N[\{(u,v)\}] - S| \geq 2|V(H)| > |V(H)| \geq |S| \geq |N[\{(u,v)\}] \cap S|,$$

which is a contradiction. Hence, $|p_{C_n}(S)| \geq 2$. $\square$

Theorem 3.2 *For $n \geq 4$ we have*

$$s(C_n \circ H) = 2|V(H)|.$$

Proof. Let $V(H) = \{v_1, \ldots, v_{|V(H)|}\}$, $V(C_n) = \{u_1, \ldots, u_n\}$ and let $u_i u_{i+1} \in E(G)$ for $i \in \{1, \ldots, n\}$, where $u_{n+1} = u_1$.

Let S be a set formed by two consecutive H-layers of graph $C_n \circ H$. It is easy to verify that S is a secure set since for every subset $X \subseteq S$ we have two possibilities, namely $|p_{C_n}(X)| \in \{1, 2\}$. For $|p_{C_n}(X)| = 1$ we have

$$|N[X] \cap S| \geq |V(H)| = |N[X] - S|$$

and for $|p_{C_n}(X)| = 2$ we have

$$|N[X] \cap S| = 2|V(H)| = |N[X] - S|.$$

Hence, $s(C_n \circ H) \leq 2|V(H)|$.

Now, suppose that S is a minimum secure set of graph $C_n \circ H$ with $|S| < 2|V(H)|$. By Lemma 3.1 we have $|p_{C_n}(S)| \geq 2$ and there exist two vertices $(u_i, v_j), (u_{i+1}, v_k) \in V(C_n \circ H)$, for some $i \in \{1, \ldots, n\}$ and some $j, k \in \{1, \ldots, |V(H)|\}$, such that $(u_i, v_j), (u_{i+1}, v_k) \in S$ as

$\langle S \rangle$ is connected. Consider the subset $X = \{(u_i, v_j), (u_{i+1}, v_k)\}$. Then $|N[X]| = 4|V(H)|$. Since $|S| < 2|V(H)|$, it follows that

$$|N[X] - S| \geq |N[X]| - |S| = 4|V(H)| - |S| > 2|V(H)| > |S| \geq |N[X] \cap S|.$$

Hence, $|N[X] \cap S| < |N[X] - S|$ for the set $X \subseteq S$ which implies that S is not a secure set, a contradiction. Therefore, $s(C_n \circ H) = |S| \geq 2|V(H)|$. □

Next we continue with description of $s(P_n \circ H)$. For this we need the following notation. For any path P_n let $V(P_n) = \{1, \ldots, n\}$ and $E(P_n) = \{i(i+1) \,|\, 1 \leq i \leq n-1\}$. For a secure set S in $P_n \circ H$ set $S^i = {}^iH \cap S$.

The value of $s(P_n \circ H)$ depends more on the structure of H. In particular, it depends on the cardinality of a minimum secure set that contains at least half of vertices of H. In the last section the result for $s(P_n \circ H)$ will be generalized. The following lemmas are needed.

Lemma 3.3 *Let S be a secure set of G. If $|N[X] \cap S| \geq |N[X] - S| + 2\ell$ for any $X \subseteq S$, then $s(G) \leq |S| - \ell$.*

Proof. Let S' be an arbitrary subset of S with $|S| - \ell$ vertices. We will prove that S' is a secure set of G. If X is an arbitrary subset of $S' \subseteq S$, then $|N[X] \cap S| \geq |N[X] - S| + 2\ell$ by assumption. Therefore $|N[X] \cap S'| \geq |N[X] \cap S| - \ell$ and $|N[X] - S'| \leq |N[X] - S| + \ell$, which implies $|N[X] \cap S'| \geq |N[X] - S'|$. □

Lemma 3.4 *Let H be an arbitrary graph, $n \geq 3$, and let S be a secure set in $P_n \circ H$ with $p_{P_n}(S) = \{1, 2\}$. If $p_H(S^1) = |V(H)| - \ell$, then $S_2 = p_H(S^2)$ is a secure set in H with the property $|N[A_2] \cap S_2| \geq |N[A_2] - S_2| + 2\ell$ for any $A_2 \subseteq S_2$.*

Proof. Suppose that there exists $A_2 \subseteq S_2$ such that $|N[A_2] \cap S_2| < |N[A_2] - S_2| + 2\ell$. Let $A = \{(2, h) \,|\, h \in A_2\} \subseteq S$. Then $|N[A] \cap S| = |V(H)| - \ell + |N[A_2] \cap S_2|$ and $|N[A] - S| = \ell + |V(H)| + |N[A_2] - S_2|$, which implies

$$\begin{aligned} |N[A] \cap S| &= |V(H)| - \ell + |N[A_2] \cap S_2| \\ &< |V(H)| - \ell + |N[A_2] - S_2| + 2\ell \\ &= \ell + |V(H)| + |N[A_2] - S_2| \\ &= |N[A] - S|, \end{aligned}$$

a contradiction. □

Note that minimal secure sets of P_n are of size 1 or 2. There are exactly two minimal secure sets of size 1, i.e. each leaf of the path is a secure set. The minimal secure sets of size 2 contain two adjacent vertices of degree 2. Clearly all minimal secure sets of P_n are tight and hence $s(P_n \circ K_t) = t$ by Theorem 2.3.

We settle all other cases for $s(P_n \circ H)$ with the next result. First, let us introduce another notation. Let $G = (V, E)$ be a graph and $x \geq 1$ an integer. Then $S_x(G)$ is a secure set of minimum cardinality containing at least x vertices of graph G. It is easy to see that for $2 \leq x \leq n$ we have $|S_x(P_n)| = |S_x(C_n)| = x$. In particular, we are interested in $S_a(H)$ for $a = \left\lceil \frac{|V(H)|}{2} \right\rceil$. While one can expect that for many graphs $S_a(H) = a$ holds, there exist graphs with $S_a(H) \geq s(G) > a$ as mentioned in the introduction, see [4]. Another way to get $S_a(H) > a$ is that $s(H) < a$ but there is no secure set of cardinality a. Some investigation into the direction, when this is not possible, is presented in [10] for global security number.

Theorem 3.5 *Let H be a non-complete graph and $n \geq 4$. For $a = \left\lceil \frac{|V(H)|}{2} \right\rceil$ we have $s(P_n \circ H) = |V(H)| + |S_a(H)|$.*

Proof. Let H be a non-complete graph, $a = \left\lceil \frac{|V(H)|}{2} \right\rceil$, and $n \geq 4$. First we construct a secure set in $P_n \circ H$ with $|V(H)| + |S_a(H)|$ vertices. Let

$$S = \{(1, h) \mid h \in V(H)\} \cup \{(2, h) \mid h \in S_a(H)\}.$$

We will prove that S is a secure set in $P_n \circ H$. Let $X \subseteq S$. If $p_{P_n}(X) = \{1, 2\}$, then

$$|N[X] \cap S| = |V(H)| + |S_a(H)| \geq |(V(H))| + (|V(H)| - |S_a(H)|) = |N[X] - S|,$$

as $|S_a(H)| \geq \frac{|V(H)|}{2}$. If $p_{P_n}(X) = \{1\}$, then

$$|N[X] \cap S| > |S_a(H)| \geq (|V(H)| - |S_a(H)|) = |N[X] - S|.$$

Finally let $p_{P_n}(X) = \{2\}$ and let $X_2 = p_H(X)$. Then

$$|N[X] \cap S| = |V(H)| + |N_H[X_2] \cap S_a(H)| \geq |V(H)| + |N_H[X_2] - S_a(H)| = |N[X] - S|$$

since $S_a(H)$ is a secure set of H.

For the converse let S be a secure set in $P_n \circ H$ with $s(P_n \circ H)$ vertices and let first $n \geq 5$. Let $i \in \{3, \ldots, n-2\}$. Suppose that there exists $h \in V(H)$ such that $(i, h) \in S$. Vertex (i, h) has $2|V(H)|$ neighbors in ${}^{i-1}H \cup {}^{i+1}H$. Therefore, there exists at least one vertex in $({}^{i-1}H \cup {}^{i+1}H) \cap S$, say $(i+1, h')$, otherwise set $\{(i, h)\} \subseteq S$ is not secure. Now, $\{(i, h), (i+1, h')\} \subseteq S$ has $4|V(H)|$ vertices in its closed neighborhood in $P_n \circ H$ which yields at least $2|V(H)| > |V(H)| + |S_a(H)|$ vertices in S, a contradiction. Notice, that the same holds if $n = 4$ and we have $(2, h), (3, h') \in S$. Hence, we may assume that S is a subset either of the first two or the last two H-layers of $P_n \circ H$. Without loss of generality we may assume that S is a subset of ${}^1H \cup {}^2H$. It is also clear that ${}^1H \cap S \neq \emptyset$ and ${}^2H \cap S \neq \emptyset$, as otherwise there exists $(x, y) \in S$ with

$$|N[\{(x, y)\}] \cap S| < |V(H)| \leq |N[\{(x, y)\}] - S|$$

since H is not a complete graph.

Let $x_i = |S \cap^i H|$ for $i \in \{1, 2\}$. As $|N[S]|$ contains $3 \cdot |V(H)|$ vertices,

$$|S| = x_1 + x_2 \geq \frac{3|V(H)|}{2}.$$

Suppose that $x_1 = |V(H)| - \ell$. Lemma 3.4 implies that $S_2 = p_H(S^2)$ is a secure set in H where $|N[A_2] \cap S_2| \geq |N[A_2] - S_2| + 2\ell$ for any $A_2 \subseteq S_2$. Therefore, it follows from Lemma 3.3 that any subset S_2' of S_2 containing $|S_2| - \ell$ vertices is a secure set in H. Let $\hat{S}^2 = \{(2, h) \mid h \in S_2'\}$. Since $x_1 + x_2 \geq \frac{3|V(H)|}{2}$ and $x_1 = |V(H)| - \ell$, it follows that

$$x_2 \geq \frac{|V(H)|}{2} + \ell.$$

Therefore $|\hat{S}^2| \geq \frac{|V(H)|}{2}$. Since S_2' is a secure set in H of size $|\hat{S}^2|$ we get

$$|S| = |V(H)| + |S_2'| \geq |V(H)| + |S_a(H)|,$$

which completes the proof. $\square$

Corollary 3.6 *If $n \geq 4$ and $m \geq 3$, then $s(P_n \circ P_m) = \lceil \frac{3m}{2} \rceil$.*

Corollary 3.7 *If $n \geq 4$ and $m \geq 4$, then $s(P_n \circ C_m) = \lceil \frac{3m}{2} \rceil$.*

4 General results

We presented the exact results for the security number of the lexicographic product of a path or a cycle and an arbitrary graph H and also the upper bound of the security number of the lexicographic product of an arbitrary graph G and a complete graph. In the case of $G \circ K_n$ and $C_n \circ H$ the upper bound is the product of the security number of the first factor and the size of the second factor. We also proved that there exist graphs where this bound is not achieved and the difference can be arbitrary large, see Corollary 2.6. On the other hand, there are many graphs G and H where the security number of $G \circ H$ is much more than the product of the security number of the first factor and the size of the second factor.

Proposition 4.1 *Let G be an arbitrary connected graph with $\delta(G) = 1$ and $|V(G)| \geq 3$. If H is an arbitrary non-complete graph, then $s(G \circ H) > s(G) \cdot |V(H)|$.*

Proof. Let S be a minimum secure set in $G \circ H$. As $s(G) = 1$, we will prove that $|S| > |V(H)|$. For the purpose of contradiction, suppose that $|S| \leq |V(H)|$. It is clear that one H-layer, say ${}^gH$, is not a secure set of $G \circ H$ since H is not complete and there exists $h \in V(H)$ with $|N_H[\{h\}]| < |V(H)|$. Therefore

$$|N[\{(g, h)\}] \cap S| < |V(H)| \leq |N[\{(g, h)\}] - S|.$$

Let X be a subset of S that contains one vertex from ${}^gH$ for every $g \in p_G(S)$. Therefore $|X| = |p_G(S)|$. Since $\langle S \rangle$ and G are connected, we have $|N[X]| \geq 3 \cdot |V(H)|$. Also, since $|S| \leq |V(H)|$, we have $|N[X] \cap S| \leq |V(H)|$ and $|N[X] - S| \geq 2 \cdot |V(H)|$, a contradiction. $\square$

The following open problem is connected with Proposition 4.1.

Problem 4.2 *For which graphs is inequality $s(G \circ H) > s(G) \cdot |V(H)|$ true? In particular, is it true if every minimum secure set of G is tight and H is a non-complete graph?*

We continue with an exact result that is a generalization of Theorem 3.5. For this let v be a vertex with $\delta(v) = 1$. Its unique neighbor, say u, is called *the support vertex* of v. The proof of the next result is identical to the proof of Theorem 3.5, only at the start of the second paragraph one needs to start with a vertex w which is not a support vertex of degree 2. Consequently, one can get even more vertices in its neighborhood as in the proof of Theorem 3.5.

Theorem 4.3 *Let H be a non-complete graph, $|V(G)| \geq 4$, and let $a = \left\lceil \frac{|V(H)|}{2} \right\rceil$. If there exists a vertex v of degree one in G with its support vertex u of degree two, then we have $s(G \circ H) = |V(H)| + |S_a(H)|$.*

Next we generalize Theorem 3.2. For this we need two adjacent vertices of degree two in G and we need to exclude the case of Theorem 4.3. We include the proof which is similar to the proof of Theorem 3.2 but contains some important differences. First a lemma.

Lemma 4.4 *Suppose that G and H are connected graphs, $|V(G)| \geq 4$, and that H is non-complete. For any secure set S in $G \circ H$ we have $|p_G(S)| \geq 2$.*

Proof. Suppose that $|p_G(S)| = 1$ for a secure set S. If $p_H(S) = V(H)$, then there exists $(u, v) \in S$ where $p_H((u, v))$ is not adjacent to all other vertices of H as H is non-complete. Since $p_G(S) = 1$, it follows

$$|N[\{(u, v)\}] - S| \geq |V(H)| = |S| > |N[\{(u, v)\}] \cap S|,$$

which is a contradiction. If $p_H(S) \subset V(H)$, then for any $(u, v) \in S$ we have

$$|N[\{(u, v)\}] - S| \geq |V(H)| > |S| \geq |N[\{(u, v)\}] \cap S|,$$

again a contradiction. Both options imply that $|p_G(S)| \geq 2$. $\square$

Theorem 4.5 *Suppose that G and H are connected graphs, $|V(H)| \geq 4$, H is non-complete, and that all support vertices in G, if they exists, have degree more than 2. If there exist two adjacent vertices of degree 2 in G, then*

$$s(G \circ H) = 2|V(H)|.$$

Proof. Let $uv \in E(G)$ where $\delta(u) = \delta(v) = 2$ and let S be the set ${}^uH \cup {}^vH$. It is easy to verify that S is a secure set, since for every subset $X \subseteq S$ we have two possibilities, namely $|p_G(X)| \in \{1, 2\}$. For $|p_G(X)| = 1$ we have

$$|N[X] \cap S| \geq |V(H)| = |N[X] - S|$$

and for $|p_G(X)| = 2$ we have

$$|N[X] \cap S| = 2|V(H)| = |N[X] - S|.$$

Hence, $s(G \circ H) \leq 2|V(H)|$.

Now, suppose that S is a minimum secure set of a graph $G \circ H$ with $|S| < 2|V(H)|$. By Lemma 4.4 we have $|p_G(S)| \geq 2$ and there exist two vertices $(g, h), (g', h') \in V(G \circ H)$, such that $(g, h), (g', h') \in S$ and $gg' \in E(G)$ as $\langle S \rangle$ is connected. Consider the subset $X = \{(g, h), (g', h')\}$. We have $|N[X]| \geq 4|V(H)|$ since every support vertex of G has degree more than 2. Since $|S| < 2|V(H)|$, it follows that

$$|N[X] - S| \geq |N[X]| - |S| > 2|V(H)| > |S| \geq |N[X] \cap S|.$$

Hence, the security condition is not true for the set $X \subseteq S$ which implies that S is not a secure set, a contradiction. Therefore, $s(G \circ H) = |S| \geq 2|V(H)|$ and we are done. $\square$

On the other hand, the product of the security number of the first factor and the size of the second factor is an upper bound for the security number of the lexicographic product in the following graphs. In particular, the following result helps if $\langle X \rangle$ is not connected.

Proposition 4.6 *Let G be a graph that contains a minimum secure set S_1 such that for any $X \subseteq S_1$, $|N[X] \cap S_1| \geq |N[X] - S_1| + k$, where k is the number of connected components of $\langle X \rangle$. Then*

$$s(G \circ H) \leq s(G) \cdot |V(H)|.$$

Proof. Let S_1 be a minimum secure set of G such that for any $X \subseteq S_1$, $|N[X] \cap S_1| \geq |N[X] - S_1| + k$, where k is the number of connected components of $\langle X \rangle$. Let $S = \{(g, h) \,|\, g \in S_1, h \in V(H)\}$. We will prove that S is a secure set in $G \circ H$. Let Y be an arbitrary subset of S and let $X = p_G(Y)$. Let $X_1, \ldots, X_k$ be connected components of $\langle X \rangle$ and $Y_i = (X_i \times V(H)) \cap Y$. If $|X_i| = 1$, then

$$|V(H)| \cdot (|N[X_i] \cap S_1| - 1) < |N[Y_i] \cap S| \leq |V(H)| \cdot |N[X_i] \cap S_1|.$$

On the other hand, if $|X_i| \geq 2$, then

$$|N[Y_i] \cap S| = |V(H)| \cdot |N[X_i] \cap S_1|.$$

Therefore,

$$|N[Y] \cap S| \geq |V(H)| \cdot (|N[X] \cap S_1| - k) \geq |V(H)| \cdot |N[X] - S_1| = |N[Y] - S|.$$

$\square$

Acknowledgment

This work has been financed by ARRS Slovenia under the grant P1-0297.

References

[1] R. C. Brigham, R. D. Dutton, S. T. Hedetniemi, Security in graphs, *Discrete Appl. Math.* 155 (13) (2007) 1708–1714.

[2] S. M. Hedetniemi, S. T. Hedetniemi, P. Kristiansen, Alliances in graphs, *J. Combin. Math. Combin. Comput.* 48 (2004) 157–177.

[3] R. D. Dutton, On a graph's security number, *Discrete Math.* 309 (13) (2009) 4443–4447.

[4] R. D. Dutton, R. Lee, R. C. Brigham, Bound on a graph's security number, *Discrete Appl. Math.* 156 (5) (2008) 695–704.

[5] I. González Yero, M. Jakovac, D. Kuziak, The security number of strong grid-like graphs, *Theoret. Comput. Sci.* 653 (2016) 1–14.

[6] R. Hammack, W. Imrich, S. Klavžar, *Handbook of Product Graphs, Second Edition*, CRC Press, Boca Raton, FL, 2011.

[7] K. Kozawa, Y. Otachi, K. Yamazaki, Security number of grid-like graphs, *Discrete Appl. Math.* 157 (11) (2009) 2555–2561.

[8] K. Jesse-Józefczyk, Monotonicity and expansion of global secure sets, *Discrete Math.* 312 (23) (2012) 3451–3456.

[9] K. Jesse-Józefczyk, Bounds on global secure sets in cactus trees, *Cent. Eur. J. Math.* 10 (3) (2012) 1113–1124.

[10] K. Jesse-Józefczyk, The possible cardinalities of global secure sets in cographs, *Theoret. Comput. Sci.* 414 (2012) 38–46.

[11] Y. Yu Ho, R. D. Dutton, Constructions of global secure sets of grid like graphs, Technical Report 01, University of Central Florida, 2010. See: *http://www.eecs.ucf.edu/techreports/*.

[12] Y. Yu Ho, R. D. Dutton, Global secure sets of grid-like graphs, *Discrete Appl. Math.* 159 (6) (2011) 490–496.