The security number of strong grid-like graphs[☆]

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Abstract

The concept of a secure set in graphs was first introduced by Brigham *et al.* in 2007 as a generalization of defensive alliances in graphs. Defensive alliances are related to the defence of a single vertex. However, in a general realistic settings, a defensive alliance should be formed so that any attack on the entire alliance or any subset of the alliance can be defended. In this sense, secure sets represent an attempt to develop a model of this situation. Given a graph $G = (V, E)$ and a set of vertices $S \subseteq V$ of G, the set S is a secure set if it can defend every attack of vertices outside of S, according to an appropriate definition of "attack" and "defence". The minimum cardinality of a secure set in G is the security number $s(G)$. In this article we obtain the security number of grid-like graphs, which are the strong products of paths and cycles (grids, cylinders and toruses). Specifically we show that for any two positive integers $m, n \geq 4$, $s(P_m \boxtimes P_n) = \min\{m, n, 8\}$, $s(P_m \boxtimes C_n) = \min\{2m, n, 16\}$ and $s(C_m \boxtimes C_n) = \min\{2m, 2n, 32\}$.

Keywords:
Secure set, security number, strong product graph, grid, cylinder, torus
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1. Introduction

Secure sets and security number in graphs were first described by Brigham *et al.* [1] while attempting to improve the well-known concepts of defensive alliances and defensive alliance number in graphs [2]. After this seminal work on secure sets in graphs, relatively few articles have been published regarding this topic. Some general results on security number are presented in [3, 4]. According to the definition in [2], defensive alliances only defend a single vertex at a given time. Nevertheless, in general models, a more efficient defensive alliance should be able to defend any attack on the entire alliance or any part of it. Studies of security in product graphs were initiated in [1], and afterwards continued in [5, 6, 7] where several bounds and closed formulaes on the security number of some grid-like graphs were given. Secure sets have also been investigated in [8, 9, 10].

We begin with some notation and terminology. In this paper $G = (V, E)$ denotes a simple graph of order n, minimum degree δ and maximum degree Δ. For a nonempty subset $W \subseteq V$ and any vertex $v \in V$, $N_W(v)$ is the set of neighbors of the vertex v in W, $N_W(v) = \{u \in W : uv \in E(G)\}$, and $\delta_W(v) = |N_W(v)|$ denotes the degree of v in W. If $W = V$, then we use the notation $N(v)$ and call it the *open neighborhood* of v. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set W is $N(W) = \bigcup_{v\in W} N(v)$ and the *closed neighborhood* of W is $N[W] = N(W) \cup W$. The subgraph induced by a set W is denoted by $\langle W\rangle$, and the complement of W is denoted by $\overline{W}$.

The definition of a secure set is based on the following rules. Consider a set of vertices S. A vertex $y \in N[S]-S$ can attack only one neighbor in S (it does not matter if y is adjacent to several vertices in S). On the other hand, a vertex $x \in S$ can defend only one vertex in $N[x] \cap S$. We now present a formal definition of secure sets according to [1].

- For any $S = \{v_1, v_2, \ldots, v_r\} \subseteq V$, an attack on S is formed by any r mutually disjoint sets $A = \{A_1, A_2, \ldots, A_r\}$, for which $A_i \subseteq N_{\overline{S}}(v_i)$, $1 \le i \le r$.
- A defence of S is formed by any r mutually disjoint sets $D = \{D_1, D_2, \ldots, D_r\}$ for which $D_i \subseteq N_S[v_i]$, $1 \le i \le r$.

- Attack A is defendable if there exists a defence D such that $|D_i| \geq |A_i|$ for $1 \leq i \leq r$.
- Set S is secure if and only if every attack on S is defendable.

The minimum cardinality of a secure set in a graph G is the *security number* and is denoted by $s(G)$. A secure set S of cardinality $s(G)$ is called a $s(G)$-set. Throughout the article we use the following characterization of secure sets.

Theorem 1. [1] *A set S is a secure set in a graph G if and only if for every $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$.*

From now on we call the expression $|N[X] \cap S| \geq |N[X] - S|$ the *security condition* for X. Studies on the security number in product graphs were initiated in [1], where the authors gave upper bounds for the Cartesian product of paths and cycles. Moreover, to prove the equality in these bounds was left as an open problem, which was solved in [5]. There was proved that for any integers $m, n \geq 4$, it follows that $s(P_m \Box P_n) = \min\{m, n, 3\}$, $s(P_m \Box C_n) = \min\{2m, n, 6\}$ and $s(C_m \Box C_n) = \min\{2m, 2n, 12\}$. Other studies of the global security number of the Cartesian product of paths and cycles where presented in [6, 7]. In the present article we prove a formula for the security number of the strong product of paths and/or cycles.

We recall that the strong product of two graphs $G = (U, E_1)$ and $H = (V, E_2)$ is the graph $G \boxtimes H$, with the vertex set $\{(a, b) : a \in U, b \in V\}$ and two vertices (a, b) and (c, d) of $U \times V$ are adjacent in $G \boxtimes H$ if and only if, either ($a = c$ and $bd \in E_2$), ($b = d$ and $ac \in E_1$), or ($ac \in E_1$ and $bd \in E_2$). The graphs G and H are called the *factors* of the product. For a vertex $a \in U$, the set of vertices $\{(a, b) : b \in V\}$ is called an *H-layer* and is denoted by ${}^aH$. Similarly, for a vertex $b \in V$, the set of vertices $\{(a, b) : a \in U\}$ is called a *G-layer* and is denoted by G^b. It is clear that the graph induced by any G-layer is isomorphic to G and, analogously, the graph induced by any H-layer is isomorphic to H. The projection of a set $W \subset U \times V$ onto G is defined by $P_G(W) = \{u \in U : (u, v) \in W\}$. Analogously, the projection of W onto H is $P_H(W) = \{v \in V : (u, v) \in W\}$. The proof of the following result is the main goal of this article.

Theorem 2. *Let $m, n \geq 2$ be two integers. Then*

(i) $s(P_m \boxtimes P_n) = \min\{m, n, 8\}$,

(ii) $s(P_m \boxtimes C_n) = \min\{2m, n, 16\}$,

(iii) $s(C_m \boxtimes C_n) = \begin{cases} 5, & \textit{if } m = n = 3, \\ \min\{2m, 2n, 32\}, & \textit{otherwise.} \end{cases}$

We split the proof of the results above in two parts (see Sections 2 and 3). Notice that the graph $C_3 \boxtimes C_3$ is isomorphic to the complete graph K_9, and from [1] we know that $s(C_3 \boxtimes C_3) = 5$.

The paper is organized as follows. Throughout the article $U = \{u_0, \ldots, u_{m-1}\}$ and $V = \{v_0, \ldots, v_{n-1}\}$ represent the vertex sets of graphs G and H of order m and n, respectively, where G and H are a path or a cycle. All the operations with the subscripts are done modulo m or n, respectively, for those cases in which the corresponding factor graph is a cycle. We only consider non-symmetrical cases.

2. Proofs of the upper bounds

Let S be a set of vertices in a graph G and let $\mathcal{UD}$ be a set of disjoint pairs of vertices $\{u, v\}$ such that $u \in S$ and $v \in N[S] - S$. The set $\mathcal{UD}$ is a *universal defence* for S, if for any attack on S, the attack of a vertex $v \in N[S] - S$ can be repelled by a vertex $u \in S$ such that $\{u, v\} \in \mathcal{UD}$.

Lemma 3. *Let $G = (V, E)$ be a graph and let $S \subset V$. If there exists a universal defence for S, then S is a secure set.*

Proof. Let $A = \{A_1, A_2, \ldots, A_r\}$ be an attack on S and let $A_i = \{v_{i1}, v_{i2}, \ldots, v_{ik_i}\}$ for every $i \in \{1, \ldots, r\}$. Since there exists a universal defence $\mathcal{UD}$ for S, every v_{ij} belongs to a pair $\{u_{ij}, v_{ij}\} \in \mathcal{UD}$, where $u_{ij} \in S$ and the vertex u_{ij} does not appear in any other pair. Thus, the set $D_i = \{u_{i1}, u_{i2}, \ldots, u_{ik_i}\}$ satisfies the condition $|D_i| \geq |A_i|$ for every $i \in \{1, \ldots, r\}$. Therefore, S is a secure set. $\square$

In the proof of the following claims we construct a universal defence for the corresponding set and then we use the lemma above.

Claim 1. *The sets ${}^{u_0}P_n$, ${}^{u_{m-1}}P_n$, $P_m{}^{v_0}$ and $P_m{}^{v_{n-1}}$ are secure sets of $P_m \boxtimes P_n$, where $m, n \geq 2$.*

Proof. First we show that the set ${}^{u_0}P_n$ is secure. The result follows from Lemma 3 and the fact that the set of disjoint pairs $\{(u_0, v_0), (u_1, v_0)\}$, $\{(u_0, v_1), (u_1, v_1)\}$,

..., $\{(u_0, v_{n-1}), (u_1, v_{n-1})\}$ is a universal defence of ${}^{u_0}P_n$. By the structure of $P_m \boxtimes P_n$, the security of ${}^{u_{m-1}}P_n$, $P_m{}^{v_0}$ and $P_m{}^{v_{n-1}}$ can be proved analogously. $\square$

Claim 2. *The sets ${}^{u_0}C_n$, ${}^{u_{m-1}}C_n$ and any two consecutive P_m-layers are secure sets of $P_m \boxtimes C_n$, where $m \geq 2$ and $n \geq 3$.*

Proof. As in the proof of Claim 1, we observe that ${}^{u_0}C_n$ and ${}^{u_{m-1}}C_n$ are secure sets. Without loss of generality, let $P_m^{v_1}$ and $P_m^{v_2}$ be two consecutive P_m-layers. The result follows from Lemma 3 and the fact that the set of disjoint pairs $\{(u_0, v_1), (u_0, v_0)\}$, $\{(u_1, v_1), (u_1, v_0)\}$, ..., $\{(u_{m-1}, v_1), (u_{m-1}, v_0)\}$, $\{(u_0, v_2), (u_0, v_3)\}$, $\{(u_1, v_2), (u_1, v_3)\}$, ..., $\{(u_{m-1}, v_2), (u_{m-1}, v_3)\}$ is a universal defence of two consecutive P_m-layers. $\square$

Since the next result can be proved similarly to Claim 2, we omit the proof.

Claim 3. *Any two consecutive C_n-layers and any two consecutive C_m-layers are secure sets of $C_m \boxtimes C_n$, where $m, n \geq 3$.*

With the claims presented above, we are able to prove the upper bounds of our main result.

Proposition 4. *For any integers $m, n \geq 2$,*

$$s(P_m \boxtimes P_n) \leq \min\{m, n, 8\}\,.$$

Proof. By Claim 1 we have that $s(P_m \boxtimes P_n) \leq \min\{m, n\}$. Assume that $m, n \geq 8$ and let $Z_{0,0}$ be the set of 8 vertices of $P_m \boxtimes P_n$ shown in Figure 1. One can see that the pairs $\{(u_0, v_1), (u_0, v_3)\}$, $\{(u_0, v_2), (u_1, v_3)\}$,

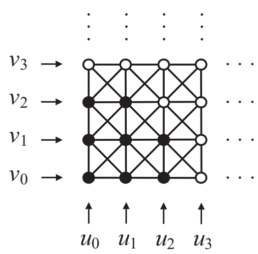


Figure 1: A secure set $Z_{0,0}$ of $P_m \boxtimes P_n$.

$\{(u_1, v_2), (u_2, v_3)\}$, $\{(u_1, v_1), (u_2, v_2)\}$, $\{(u_2, v_1), (u_3, v_2)\}$, $\{(u_1, v_0), (u_3, v_1)\}$ and $\{(u_1, v_0), (u_3, v_0)\}$ form a universal defence of $Z_{0,0}$. Hence, by Lemma 3, $Z_{0,0}$ is a secure set. $\square$

Notice that the cylinder $P_m \boxtimes C_n$ can be obtained from the grid $P_m \boxtimes P_n$ by adding some edges between vertices of the ${P_m}^{v_0}$-layer and the ${P_m}^{v_{n-1}}$-layer (see Figure 2).

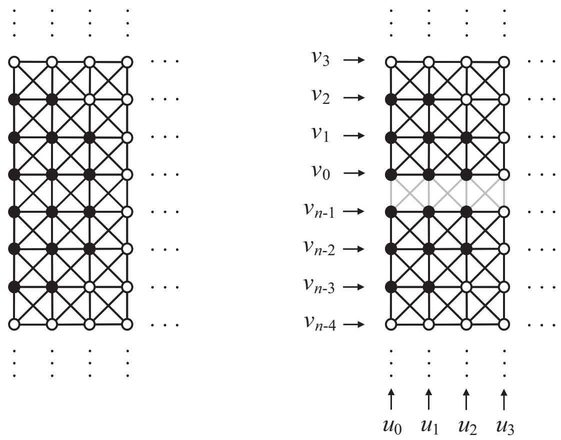


Figure 2: A secure set S of $P_m \boxtimes C_n$ with $|S| = 16$ and the edges added to the graph $P_m \boxtimes P_n$ to obtain $P_m \boxtimes C_n$.

This observation allows us to find a universal defence of set S presented in Figure 2, similarly as we found it for $Z_{0,0}$ in the proof of Proposition 4. The next result follows from the security of S and Claim 2.

Proposition 5. *For any integers $m, n \geq 2$,*

$$s(P_m \boxtimes C_n) \leq \min\{2m, n, 16\}\,.$$

We can also indicate a secure set of cardinality 32 in a sufficiently large strong product of cycles (see Figure 3). The existence of such a set and Claim 3 imply the following proposition.

Proposition 6. *For any integers $m \geq 3$ and $n \geq 4$,*

$$s(C_m \boxtimes C_n) \leq \min\{2m, 2n, 32\}\,.$$

3. Proofs of the lower bounds

Let S be a minimum secure set of $G \boxtimes H$, where G and H are a path or a cycle. For $i \in \{0, 1, \ldots, m-1\}$ we define $X_i = {}^{u_i}H \cap S$. Let $d =$

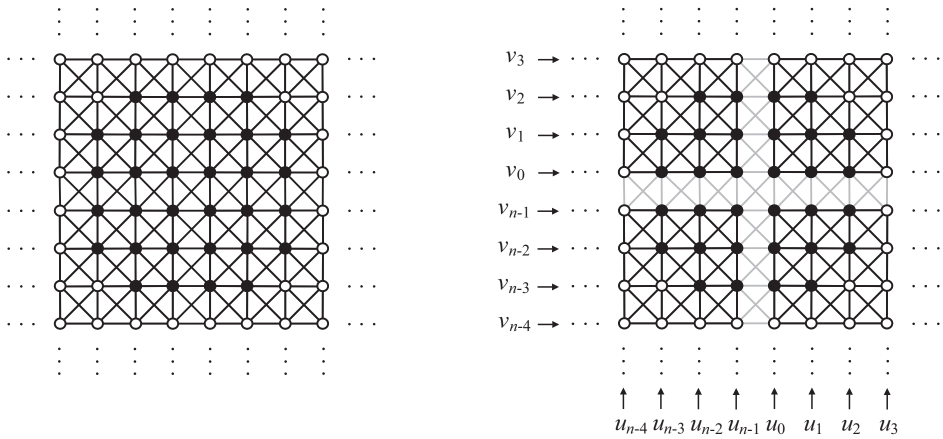


Figure 3: A secure set S of $C_m \boxtimes C_n$ with $|S| = 32$ and the edges added to the graph $P_m \boxtimes P_n$ to obtain $C_m \boxtimes C_n$.

$\min\{i : 0 \leq i \leq m-1,\ X_i \neq \emptyset\}$, $f = \max\{i : 0 \leq i \leq m-1,\ X_i \neq \emptyset\}$ and $r = f - d + 1$. Similarly, we define $Y_j = G^{v_j} \cap S$ for $j \in \{0, 1, \ldots, n-1\}$. Let $g = \min\{j : 0 \leq j \leq n-1,\ Y_j \neq \emptyset\}$, $h = \max\{j : 0 \leq j \leq n-1,\ Y_j \neq \emptyset\}$ and $t = h - g + 1$.

From now on, we use the letters d, f, g, h, r, t (and their corresponding meanings) defined in the paragraph above. If at least one graph in the strong product is a cycle, then we can assume that ($d = 0$ and $f = r - 1$) or ($g = 0$ and $h = t - 1$). Since any minimum secure set is connected [1], it follows that for every $i \in \{d, \ldots, f\}$, $X_i \neq \emptyset$ and for every $j \in \{g, \ldots, h\}$, $Y_j \neq \emptyset$.

Given the vertices $u_i, u_j \in U$, $0 \leq i < i+1 < j \leq m-1$, and $v_k, v_l \in V$, $0 \leq k < k+1 < l \leq n-1$ of a graph G and H, respectively, we call the set of vertices $R(i,j,k,l) \subseteq U \times V$ defined by

$$R(i,j,k,l) = \{(u_a, v_b) \in U \times V : i \leq a \leq j,\ k \leq b \leq l\}$$

a *rectangle* in $G \boxtimes H$. Notice that any rectangle has nonempty intersection with at least three G-layers and at least three H-layers. According to a rectangle $R(i,j,k,l)$, for any $\alpha \in \{i,j\}$ and any $\beta \in \{k,l\}$, a *corner-like set* $C(\alpha, \beta)$ of a set $S \subseteq R(i,j,k,l) - \{(u_i, v_k), (u_i, v_l), (u_j, v_k), (u_j, v_l)\}$ such that $S \cap {}^{u_\alpha}H \neq \emptyset$ and $S \cap G^{v_\beta} \neq \emptyset$, is formed by exactly two vertices, one in X_α and the other in Y_β, being the closest vertices to (u_α, v_β) (see Figure 4). If a corner-like set contains two neighbors of the corner, then we call it *simple*. A simple corner-like set is for instance $C(i,k)$ in Figure 4.

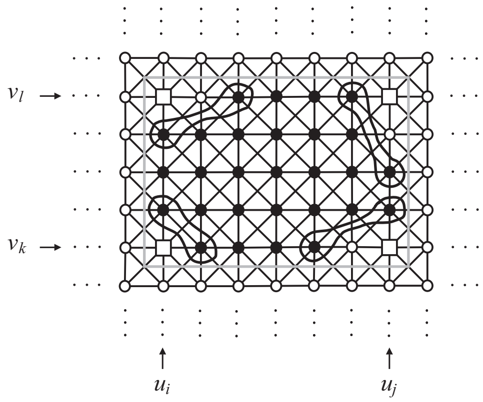


Figure 4: The rectangle $R(i, j, k, l)$ and the four corner-like sets $C(i, k)$, $C(i, l)$, $C(j, k)$ and $C(j, l)$, where $C(i, k)$ is simple.

Given a rectangle $R(i, j, k, l)$ in $G \boxtimes H$ and a set $W \subseteq R(i, j, k, l)$, we define the following sets.

- $\mathcal{LB}_W$ — the left border of W — is the set of vertices $(u_a, v_b) \in W$ such that $(u_\alpha, v_b) \notin W$ for every $\alpha \in \{i, \ldots, a-1\}$.
- $\mathcal{RB}_W$ — the right border of W — is the set of vertices $(u_a, v_b) \in W$ such that $(u_\alpha, v_b) \notin W$ for every $\alpha \in \{a+1, \ldots, j\}$.
- $\mathcal{UB}_W$ — the upper border of W — is the set of vertices $(u_a, v_b) \in W$ such that $(u_a, v_\beta) \notin W$ for every $\beta \in \{k, \ldots, b-1\}$.
- $\mathcal{BB}_W$ — the bottom border of W — is the set of vertices $(u_a, v_b) \in W$ such that $(u_a, v_\beta) \notin W$ for every $\beta \in \{b+1, \ldots, l\}$.

Figure 5 shows an example of the sets defined above. In addition we say that the union of all these sets is the *border* of W, and denote it by $\mathcal{BD}$. Notice that $\mathcal{LB}_W \cap \mathcal{UB}_W \neq \emptyset$, $\mathcal{LB}_W \cap \mathcal{BB}_W \neq \emptyset$, $\mathcal{RB}_W \cap \mathcal{UB}_W \neq \emptyset$ and $\mathcal{RB}_W \cap \mathcal{BB}_W \neq \emptyset$. Also, $|\mathcal{LB}_W| = |\mathcal{RB}_W|$ and $|\mathcal{UB}_W| = |\mathcal{BB}_W|$.

Lemma 7. *Let W be a subset of a rectangle that contains at most $n-2$ columns and at most $m-2$ rows of $C_m \boxtimes C_n$, where $m, n \geq 4$. Then $|N[W] - W| \geq |\mathcal{LB}_W| + |\mathcal{RB}_W| + |\mathcal{UB}_W| + |\mathcal{BB}_W| + 4$.*

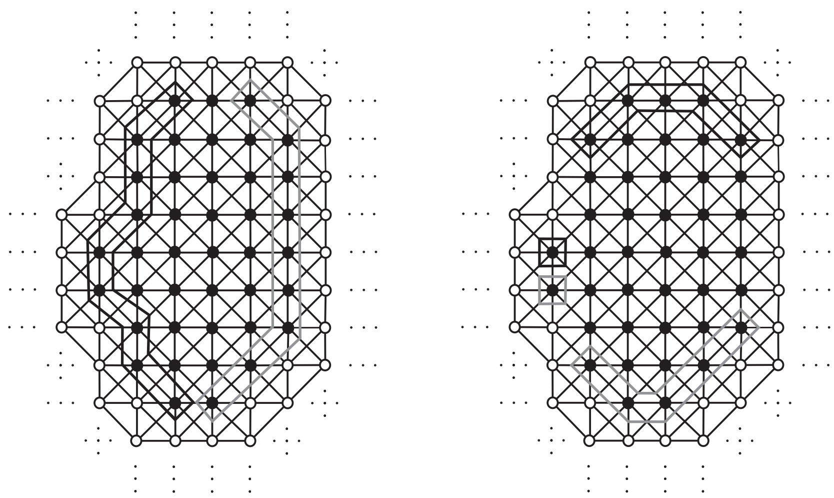


Figure 5: The sets $\mathcal{LB}_W$ (black polygon - left side), $\mathcal{RB}_W$ (gray polygon - left side), $\mathcal{UB}_W$ (black polygons - right side) and $\mathcal{BB}_W$ (gray polygons - right side).

Proof. Let $\mathcal{LB}'_W = \mathcal{LB}_W \cap (\mathcal{UB}_W \cup \mathcal{BB}_W)$, $\mathcal{RB}'_W = \mathcal{RB}_W \cap (\mathcal{UB}_W \cup \mathcal{BB}_W)$, $\mathcal{UB}'_W = \mathcal{UB}_W \cap (\mathcal{LB}_W \cup \mathcal{RB}_W)$ and $\mathcal{BB}'_W = \mathcal{BB}_W \cap (\mathcal{LB}_W \cup \mathcal{RB}_W)$. Hence,

$$\begin{aligned}
|N[W] - W| &\geq |\mathcal{LB}_W - \mathcal{LB}'_W| + 2|\mathcal{LB}'_W| + |\mathcal{RB}_W - \mathcal{RB}'_W| + 2|\mathcal{RB}'_W| + \\
&\quad + |\mathcal{UB}_W - \mathcal{UB}'_W| + |\mathcal{BB}_W - \mathcal{BB}'_W| + 4 \\
&= |\mathcal{LB}_W| + |\mathcal{LB}'_W| + |\mathcal{RB}_W| + |\mathcal{RB}'_W| + |\mathcal{UB}_W| - |\mathcal{UB}'_W| + \\
&\quad + |\mathcal{BB}_W| - |\mathcal{BB}'_W| + 4 \\
&= |\mathcal{LB}_W| + |\mathcal{LB}_W \cap \mathcal{UB}_W| + |\mathcal{LB}_W \cap \mathcal{BB}_W| + |\mathcal{RB}_W| + \\
&\quad + |\mathcal{RB}_W \cap \mathcal{UB}_W| + |\mathcal{RB}_W \cap \mathcal{BB}_W| + |\mathcal{UB}_W| + \\
&\quad - |\mathcal{UB}_W \cap \mathcal{LB}_W| - |\mathcal{UB}_W \cap \mathcal{RB}_W| + |\mathcal{BB}_W| - |\mathcal{BB}_W \cap \mathcal{LB}_W| + \\
&\quad - |\mathcal{BB}_W \cap \mathcal{RB}_W| + 4 \\
&\geq |\mathcal{LB}_W| + |\mathcal{RB}_W| + |\mathcal{UB}_W| + |\mathcal{BB}_W| + 4,
\end{aligned}$$

and the proof is completed. □

Lemma 8. *Let W be a subset of a rectangle that contains at most $n-1$ columns and at most $m-2$ rows of $P_m \boxtimes C_n$, where $m \geq 3$ and $n \geq 4$. Then $|N[W] - W| \geq |\mathcal{RB}_W| + |\mathcal{UB}_W| + |\mathcal{BB}_W| + 2$.*

Proof. The proof is similar to that of Lemma 7. Let $\mathcal{RB}'_W = \mathcal{RB}_W \cap (\mathcal{UB}_W \cup$

$\mathcal{BB}_W$), $\mathcal{UB}'_W = \mathcal{UB}_W \cap \mathcal{RB}_W$ and $\mathcal{BB}'_W = \mathcal{BB}_W \cap \mathcal{RB}_W$. Hence,

$$\begin{aligned}
|N[W] - W| &\geq |\mathcal{RB}_W - \mathcal{RB}'_W| + 2|\mathcal{RB}'_W| + |\mathcal{UB}_W - \mathcal{UB}'_W| + |\mathcal{BB}_W - \mathcal{BB}'_W| + 2 \\
&= |\mathcal{RB}_W| + |\mathcal{RB}'_W| + |\mathcal{UB}_W| - |\mathcal{UB}'_W| + |\mathcal{BB}_W| - |\mathcal{BB}'_W| + 2 \\
&= |\mathcal{RB}_W| + |\mathcal{RB}_W \cap \mathcal{UB}_W| + |\mathcal{RB}_W \cap \mathcal{BB}_W| + |\mathcal{UB}_W| - |\mathcal{UB}_W| + \\
&\quad - |\mathcal{UB}_W \cap \mathcal{RB}_W| + |\mathcal{BB}_W| - |\mathcal{BB}_W| - |\mathcal{BB}_W \cap \mathcal{RB}_W| + 2 \\
&\geq |\mathcal{RB}_W| + |\mathcal{UB}_W| + |\mathcal{BB}_W| + 2,
\end{aligned}$$

which completes the proof. □

We say that a partial structure of a secure set S is given by the set $PD = A \cup B$ where $A \subseteq S$ and $B = N[A] - S$. In the auxiliary figures, black and white vertices belong to A and $V - S$, respectively, and any gray vertex may be a member of either A or B.

Lemma 9. *If a set $S \subseteq U \times V$ is a secure set of $C_m \boxtimes C_n$ or $P_m \boxtimes C_n$, where $m, n \geq 3$, then the partial structures of S presented in Figure 6 are forbidden.*

Proof. Since the maximum degree of vertices in $C_m \boxtimes C_n$ or $P_m \boxtimes C_n$ is 8, if a set $S \subseteq U \times V$ contains a vertex that has five neighbors which are not in S, then by Theorem 1 S is not secure. Thus, the partial structures presented in Figure 6 (a) and (b) are forbidden.

In the partial structure presented in Figure 6 (c), we have that $|N[A] \cap S| \leq 10$ while $|B| \geq 12$, so this partial structure is also forbidden. Similar situations are in the partial structures illustrated in Figure 6 (d) and (f). For the case of Figure 6 (e), the set W, surrounded by a polygon, satisfies $|N[W] \cap S| \leq 8$ and $|N[W] - S| \geq 9$, which makes this partial structure also forbidden. □

The following lemma presents lower bounds for secure sets that contain at least m or n vertices.

Lemma 10. *Let $m, n \geq 4$ be two integers and let S be a $s(C_m \boxtimes C_n)$-set. If $|P_{C_m}(S)| = m$ or $|P_{C_n}(S)| = n$, then $s(C_m \boxtimes C_n) \geq 2n$ or $s(C_m \boxtimes C_n) \geq 2m$.*

Proof. Suppose that $s(C_m \boxtimes C_n) < 2n$ and $s(C_m \boxtimes C_n) < 2m$. Assume $|P_{C_m}(S)| = m$. Since S is connected, $X_i \neq \emptyset$ for every $i \in \{0, \ldots, m-1\}$. If $|X_i| \geq 2$ for every $i \in \{1, \ldots, m-1\}$, then $|S| \geq 2m$, which is a contradiction.

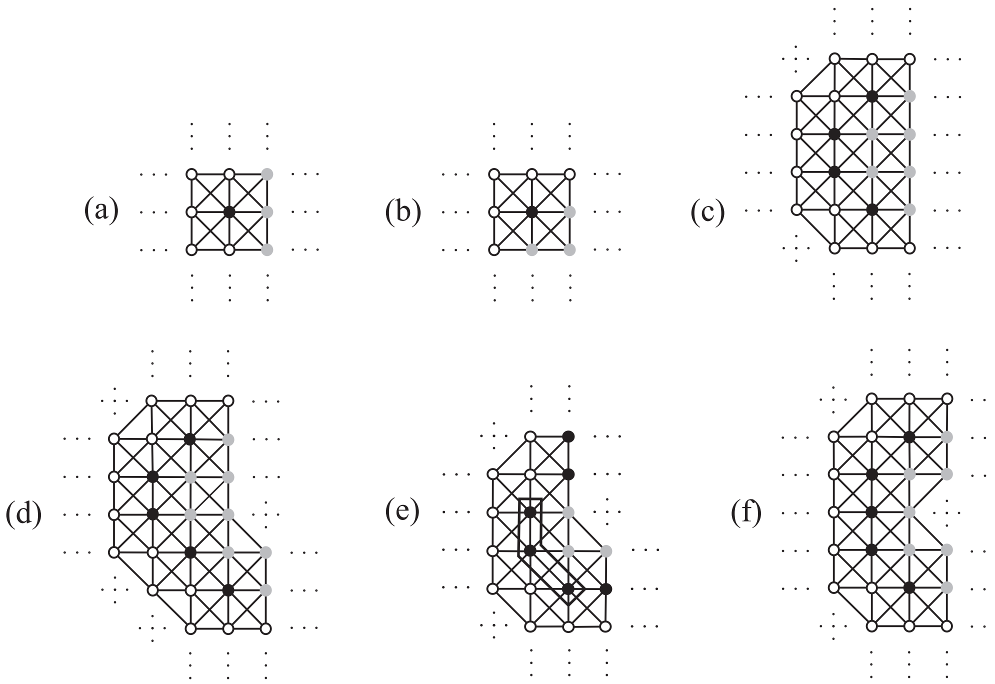


Figure 6: The partial structures which are forbidden for a secure set S.

If $|(N[X_i]-S)\cap {}^{u_i}C_n| \geq 2$ for every $i \in \{0,\ldots,m-1\}$, then $|(N[S]-S)| \geq 2m$, which contradicts the security of S. So, there exists a ${}^{u_j}C_n$-layer such that at most one vertex of it does not belong to S, and there is exactly one such layer (otherwise $|S| \geq 2m$ or $|S| \geq 2n$ since $m,n \geq 4$). Thus $|S| \geq m+n-2$. If $|S| = m+n-2$ or $|S| = m+n-1$, then for the vertex $(u_{j+2}, v_\alpha) \in S$, we have $|(N[\{(u_{j+2}, v_\alpha)\}] \cap S)| \leq 4$ and $|(N[\{(u_{j+2}, v_\alpha)\}] - S)| \geq 5$, a contradiction. If $|P_{C_n}(S)| = n$, then a similar analysis as above completes the proof. □

Similarly as above, we can prove the following lemma.

Lemma 11. *Let $m,n \geq 4$ be two integers and let S be a $s(P_m \boxtimes C_n)$-set. If $|P_{P_m}(S)| = m$, then $s(P_m \boxtimes C_n) \geq 2m$.*

In our next two results we give more properties of secure sets in $C_m \boxtimes C_n$ and $P_m \boxtimes C_n$.

Lemma 12. *If a set $S \subseteq U \times V$ is a $s(C_m \boxtimes C_n)$-set, $m,n \geq 3$, such that $|S| < \min\{2m, 2n\}$, then $|X_0|, |X_{r-1}| \geq 2$, $|Y_0|, |Y_{t-1}| \geq 2$ and $6 \leq r,t \leq 8$.*

Proof. If $r = m$ or $t = n$, then by Lemma 10, $|S| \geq 2m$ or $|S| \geq 2n$, which is a contradiction. So, $r < m$ and $t < n$. If $|X_0| = 1$, then S contains the partial structure presented in Figure 6 (a), which means that S is not secure. Thus, $|X_0| \geq 2$. According to the symmetry of $C_m \boxtimes C_n$, we have that $|X_{r-1}| \geq 2$, $|Y_0| \geq 2$ and $|Y_{t-1}| \geq 2$.

Since S does not contain the forbidden partial structures appearing in Figure 6 (a) and (b), we have that $t \geq |X_0| + 2$. First, suppose that $|X_0| = 2$. If $t = 4$, then S contains the partial structure (c) presented in Figure 6. So, S is not secure. If $t = 5$, then S contains the partial structures presented in Figure 6 (b) or (d), which are forbidden. Hence, if $|X_0| = 2$, then $t \geq 6$. Now suppose that $|X_0| = 3$. If $t = 5$, then S contains the partial structure presented in Figure 6 (f). Thus, S is not secure. Hence, $t \geq 6$. Finally, if $|X_0| \geq 4$, then $t \geq 6$. According to the symmetry of $C_m \boxtimes C_n$, we obtain that $r \geq 6$.

If $r = m - 1$, then $|S| = |X_0| + |X_1| + |X_{m-3}| + |X_{m-2}| + \sum_{i=2}^{m-4} |X_i| \geq 12 + \sum_{i=2}^{m-4} |X_i|$. We will show that $\sum_{i=2}^{m-4} |X_i| \geq 2(m-5)$. Suppose on the contrary, that $\sum_{i=2}^{m-4} |X_i| < 2(m-5)$. Hence, there exists at least one X_j, $2 \leq j \leq m-4$, such that $|X_j| = 1$ (notice that $|X_j| \neq 0$ since S is connected). We shall first show that $j \neq 2$ and $j \neq m-4$. If $j = 2$, then there exists a vertex $(u_1, v_q) \in X_1$ such that $|N[(u_1, v_q)] \cap S| \leq 3$ and $|N[(u_1, v_q)] - S| \geq 6$, a contradiction. Thus, $j \neq 2$ and, similarly, we get $j \neq m-4$. Now, observe that it follows $|X_{j-1}| \geq 2$ or $|X_{j+1}| \geq 2$, otherwise $|N[X_j] \cap S| = 3$ and $|N[X_j] - S| = 6$, which is not possible. If $|X_{j+1}| = 2$, then $|X_{j+2}| \geq 3$, otherwise $|N[X_{j+1}] \cap S| = 5$ and $|N[X_{j+1}] - S| = 7$, which is again not possible. Notice also that $|X_{j+3}| \geq 3$. If not, then $|N[X_{j+2}] \cap S| = 7$ and $|N[X_{j+2}] - S| = 8$.

As a consequence, for every X_j with $|X_j| = 1$, there exists $X_{j'}$ with $|X_{j'}| \geq 3$ such that, if $|X_j| = |X_l| = 1$, $l \neq j$, then $X_{j'} \neq X_{l'}$. Let $I \subset \{3, \ldots, m-5\}$ be such that for every $j \in I$, $|X_j| = 1$. Hence,

$$\begin{aligned}
\sum_{i=2}^{m-4} |X_i| &= \sum_{i \in I} |X_i| + \sum_{i \notin I, 2 \leq i \leq m-4} |X_i| \\
&\geq 2|I| + \sum_{i \notin I, 2 \leq i \leq m-4} 2 \\
&= 2|I| + 2(m-5-|I|) \\
&= 2(m-5),
\end{aligned}$$

which is a contradiction. Thus, $|S| \geq 12 + \sum_{i=2}^{m-4} |X_i| \geq 12 + 2(m-5) = 2m+2$, which is a contradiction. Similarly, we can show that $t \neq n-1$, since then $|S| \geq 2n+2$. Therefore, $r < m-1$ and $t < n-1$.

Now, if $r \geq 9$ or $t \geq 9$, then by Lemma 7, we obtain $|N[S] - S| \geq |\mathcal{LB}_S| + |\mathcal{RB}_S| + |\mathcal{UB}_S| + |\mathcal{BB}_S| + 4 \geq 34$, which is a contradiction with Proposition 6. □

Lemma 13. *If a set $S \subseteq U \times V$ is a $s(P_m \boxtimes C_n)$-set, $m, n \geq 3$, such that $|S| < \min\{2m, n\}$, then ($|X_d| \geq 2$ or $|X_f| \geq 2$), $3 \leq r \leq 4$ and $6 \leq t \leq 8$.*

Proof. Let S be a $s(P_m \boxtimes C_n)$-set such that $|S| < n$. Since t is the number of C_n-layers that have a nonempty intersection with S, $t \neq n$. Moreover by Lemma 11, $r \neq m$. Hence, we may assume $r < m$ and $t < n$. If $d \neq 0$ and $|X_d| = 1$, then S contains the partial structure presented in Figure 6 (a), which means that S is not secure. Thus, $|X_d| \geq 2$. Analogously, if $f \neq m-1$ and $|X_f| = 1$, then S is not secure. So, $|X_f| \geq 2$.

If $d \neq 0$ and $f \neq m-1$, then, similarly to the proof of Lemma 12 we can show that $t \geq 6$. If either $d = 0$ or $f = m-1$, say $d = 0$, then $f - d + 1 = r \geq 2$. Suppose that $r = 2$. Since $|S| < n$, there exists $j \in \{0, \ldots, n-1\}$ such that $(u_0, v_j), (u_1, v_j) \notin S$. Now, let $i \in \{0, \ldots, n-1\}$ be such that $S \cap \{(u_0, v_i), (u_1, v_i)\} \neq \emptyset$ and $(u_0, v_{i+1}), (u_1, v_{i+1}) \notin S$. If $(u_0, v_i), (u_1, v_i) \in S$, then we have a contradiction with Lemma 9 (Figure 6 (b)). So, either $(u_0, v_i) \notin S$ or $(u_1, v_i) \notin S$. If $(u_0, v_i) \notin S$, then $(u_1, v_i) \in S$, $|N[(u_1, v_i)] \cap S| \leq 3$ and $|N[(u_1, v_i)] - S| \geq 6$, which is not possible. So, $(u_0, v_i) \in S$ and $(u_1, v_i) \notin S$. If $(u_1, v_{i-1}) \notin S$, then $|N[(u_0, v_i)] \cap S| \leq 2$ and $|N[(u_0, v_i)] - S| \geq 4$, a contradiction. Hence, $(u_1, v_{i-1}) \in S$ and we have $|N[\{(u_0, v_i), (u_1, v_{i-1})\}] \cap S| \leq 5$ and $|N[\{(u_0, v_i), (u_1, v_{i-1})\}] - S| \geq 6$. So, the security condition is not satisfied for the set $\{(u_0, v_i), (u_1, v_{i-1})\}$. Thus, $r \geq 3$.

Since $d \neq 0$ or $f \neq m-1$, say $d \neq 0$, we have that $|X_d| \geq 2$. From the forbidden partial structures shown in Figure 6 (a) and (b), we obtain $t \geq |X_d| + 2$. Hence, if $|X_d| \geq 4$, then $t \geq 6$. Thus, suppose that $|X_d| = 2$. If $t = 4$, then S contains the partial structure (c) presented in Figure 6. Therefore, S is not secure. If $t = 5$, then S contains the partial structures in Figure 6 (b) or (d), which are forbidden, or S contains the partial structure in Figure 7. In the partial structure presented in Figure 7, we have that $|N[A] \cap S| \leq 11$ while $|B| \geq 13$. So, the set S containing this partial structure is not secure. Hence, if $|X_d| = 2$, then $t \geq 6$. Moreover, if $|X_d| = 3$,

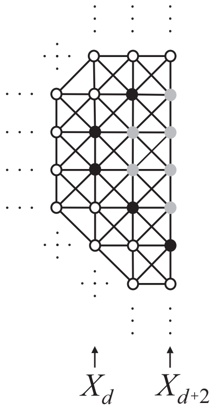


Figure 7: The partial structure of S.

then $t \geq 5$. If $t = 5$, then S contains the partial structure presented in Figure 6 (f). Thus, S is not secure. Therefore, $t \geq 6$.

If $r \geq 5$ or $t \geq 9$, then by Lemma 8 (for the symmetrical case), $|N[S] - S| \geq |\mathcal{LB}_S| + |\mathcal{UB}_S| + |\mathcal{BB}_S| + 2 \geq 18$ or $|N[S] - S| \geq |\mathcal{LB}_S| + |\mathcal{UB}_S| + |\mathcal{BB}_S| + 2 \geq 17$, respectively, and this is a contradiction with Proposition 5. $\square$

In the next result we present lower bounds on the security number of $C_m \boxtimes C_n$.

Proposition 14. *For any integers $m \geq 3$ and $n \geq 4$,*

$$s(C_m \boxtimes C_n) \geq \min\{2m, 2n, 32\}.$$

Proof. Suppose that $s(C_m \boxtimes C_n) < \min\{2m, 2n, 32\}$ and let S be a $s(C_m \boxtimes C_n)$-set. Consider the sets X_α, $\alpha \in \{0, \dots, m-1\}$ and Y_β, $\beta \in \{0, \dots, n-1\}$. First we assume that $m = 3$ and $n \geq 4$. Hence $\min\{2m, 2n, 32\} = 6$. However, we can check that every set of cardinality less than or equal to five has a neighborhood of cardinality greater than or equal to seven.

From now on we consider the case $m \geq 4$ and $n \geq 4$. If there exists a set X_i such that $|X_i| = n$, then by Lemma 10, $s(C_m \boxtimes C_n) \geq 2m$ or $s(C_m \boxtimes C_n) \geq 2n$, which is a contradiction. Similarly, we obtain a contradiction, if there exists a set Y_k such that $|Y_k| = m$.

Thus, $|X_\alpha| < n$ and $|Y_\beta| < m$ for every $\alpha \in \{0, \dots, m-1\}$ and $\beta \in \{0, \dots, n-1\}$. Hence, without loss of generality assume that S is a subset of some rectangle $R(0, r-1, 0, t-1)$. For every $j \notin \{0, \dots, r-1\}$, $X_j = \emptyset$, and for every $l \notin \{0, \dots, t-1\}$, $Y_l = \emptyset$. Notice that by Lemma 12, $6 \leq r, t \leq 8$.

If $r, t \geq 7$ or ($r = 6$ and $t \geq 8$) or ($r \geq 8$ and $t = 6$), then by Lemma 7, $|N[S] - S| \geq 32$. As a consequence, without loss of generality, assume that $6 \leq r \leq 7$ and $t = 6$. Consider the corner-like sets of a set S in the rectangle $R(0, r-1, 0, t-1)$.

(i) If $(u_0, v_a), (u_0, v_b) \in X_0$ where $a = \min\{l \in \{0, \dots, t-1\} : (u_0, v_l) \in X_0\}$ and $b = \max\{l \in \{0, \dots, t-1\} : (u_0, v_l) \in X_0\}$, then $a \geq 1$, $b \leq t-2$ and $(u_1, v_{a-1}), (u_1, v_{b+1}) \in X_1$.

(ii) If $(u_{r-1}, v_p), (u_{r-1}, v_z) \in X_{r-1}$ with $p = \min\{l \in \{0, \dots, t-1\} : (u_{r-1}, v_l) \in X_{r-1}\}$ and $z = \max\{l \in \{0, \dots, t-1\} : (u_{r-1}, v_l) \in X_{r-1}\}$, then $p \geq 1$, $z \leq t-2$ and $(u_{r-2}, v_{p-1}), (u_{r-2}, v_{z+1}) \in X_{r-2}$.

(iii) If $(u_a, v_0), (u_b, v_0) \in Y_0$ where $a = \min\{l \in \{0, \dots, r-1\} : (u_l, v_0) \in Y_0\}$ and $b = \max\{l \in \{0, \dots, r-1\} : (u_l, v_0) \in Y_0\}$, then $a \geq 1$, $b \leq r-2$ and $(u_{a-1}, v_1), (u_{b+1}, v_1) \in Y_1$.

(iv) If $(u_p, v_{t-1}), (u_z, v_{t-1}) \in Y_{t-1}$ with $p = \min\{l \in \{0, \dots, r-1\} : (u_l, v_{t-1}) \in Y_{t-1}\}$ and $z = \max\{l \in \{0, \dots, r-1\} : (u_l, v_{t-1}) \in Y_{t-1}\}$, then $p \geq 1$, $z \leq r-2$ and $(u_{p-1}, v_{t-2}), (u_{z+1}, v_{t-2}) \in Y_{t-2}$.

It follows from the observations above that $(u_0, v_0), (u_0, v_{t-1}), (u_{r-1}, v_0), (u_{r-1}, v_{t-1})$, $(u_0, v_0), (u_{r-1}, v_0), (u_0, v_{t-1}), (u_{r-1}, v_{t-1}) \notin S$. In the following cases we assume $6 \leq r \leq 7$ and $t = 6$. Moreover, we avoid the forbidden partial structures presented in Figure 6. Also, based on the existence of simple corner-like sets, we consider the possible borders of S. Note that in the auxiliary figures, vertices in the polygons drawn in the center are not adjacent to any vertex in $\mathcal{BD}$.

Case 1: No corner-like set of S is simple. We observe that only two borders of S presented in Figure 8 are possible. In Figure 8 (a), $|N[\mathcal{BD}] \cap S| \leq 24$ and $|N[\mathcal{BD}] - S| \geq 28$ and, in Figure 8 (b), $|N[\mathcal{BD}] \cap S| \leq 28$ and $|N[\mathcal{BD}] - S| \geq 30$, which are both contradictions.

Case 2: Only one corner-like set of S is simple. Hence, the only possible borders of S are drawn in Figure 9. For the case of Figure 9 (a), we have $|N[\mathcal{BD}] \cap S| \leq 26$ and $|N[\mathcal{BD}] - S| \geq 28$. Moreover, in Figure 9 (b), $|N[\mathcal{BD}] \cap S| \leq 29$ and $|N[\mathcal{BD}] - S| \geq 30$. Both cases are contradictions.

Case 3: Exactly two corner-like sets of S are simple. In this case, we consider three possibilities for the border of S according to positions of those two simple corner-like sets.

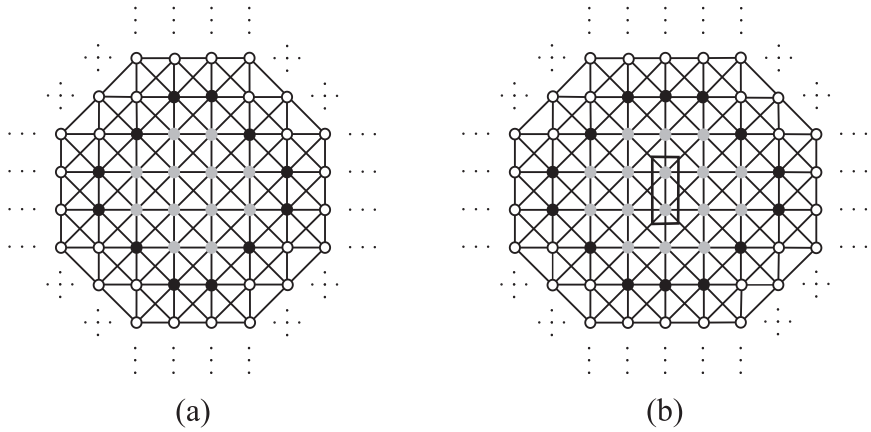


Figure 8: Possible borders of S with no simple corner-like sets.

Subcase 3.1: Both simple corner-like sets have a nonempty intersection with either X_0 or X_{r-1}. So, only the borders drawn in Figure 10 are possible for S. In Figure 10 (a), $|N[\mathcal{BD}] \cap S| \leq 26$ and $|N[\mathcal{BD}] - S| \geq 28$, a contradiction. Now, consider the $\mathcal{BD}$ shown in Figure 10 (b). Since $|N[\mathcal{BD}] - S| \geq 30$, all gray vertices adjacent to vertices in $\mathcal{BD}$ must be in S. Then, $|N[\mathcal{BD}] \cap S| = 30$ and $|N[\mathcal{BD}] - S| = 30$. Thus, from the vertices in the square drawn in the center, which are not adjacent to any vertex in $\mathcal{BD}$, at most one can belong to S, since $|S| \leq 31$ by assumption. Hence, without loss of generality we can assume that the gray vertex inside the lower triangle drawn in Figure 10 (b) has a neighbour inside the square that does not belong to S. Then, for the set of vertices inside the triangle the security condition is not satisfied, a contradiction.

Subcase 3.2: Both simple corner-like sets have a nonempty intersection with either Y_0 or Y_{t-1}. Hence, S can only have the border that is shown in Figure 11. In this case, the security condition is not violated only if all gray vertices that are adjacent to the vertices in the border belong to S. Now, $|N[\mathcal{BD}] \cap S| = |N[\mathcal{BD}] - S| = 30$, thus only one of four vertices inside the polygon can belong to S. However, similarly as in Subcase 3.1 we can show that both gray vertices that are inside the triangles must have a neighbour inside the polygon that belongs to S, a contradiction.

Subcase 3.3: Only one of the two simple corner-like sets have a nonempty intersection with X_0. In this case, only the borders drawn in Figure 12 can appear in S. In Figure 12 (a), $|N[\mathcal{BD}] \cap S| \leq 26$ and $|N[\mathcal{BD}] - S| \geq 28$, a

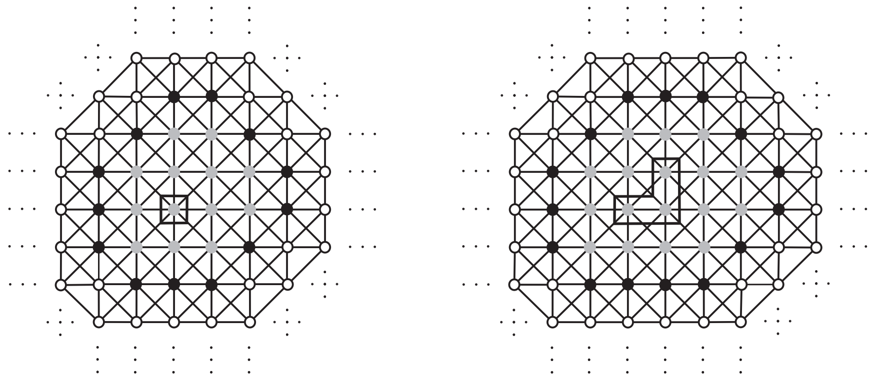


Figure 9: Possible borders of S with exactly one simple corner-like set.

contradiction. Now consider the border of the set S presented in Figure 12 (b). Since $|N[\mathcal{BD}] - S| \geq 30$, all gray vertices adjacent to vertices in $\mathcal{BD}$ must be in S. Then, $|N[\mathcal{BD}] \cap S| = 30 = |N[\mathcal{BD}] - S|$. Now, since $|S| \leq 31$ we observe that only one vertex that is inside the central polygon can belong to S. This leads us to a contradiction with the security condition that is not satisfied for the vertices inside one of the triangles.

Case 4: Exactly three corner-like sets of S are simple. In this case, S can only have one of the borders illustrated in Figure 13. However, the security condition is not satisfied for the border (a). So, let us consider the border (b). We can check that $|N[\mathcal{BD}] \cap S| \geq |N[\mathcal{BD}] - S|$ only if all gray vertices in $N(\mathcal{BD})$ are in S. Then, $|N[\mathcal{BD}] \cap S| = 31$. It follows that no vertex inside the central polygon belongs to S. Let T be the set of vertices inside the triangle. Then, we have $|N[T] \cap S| = 7$ and $|N(T) - S| = 8$, a contradiction.

Case 5: All corner-like sets of S are simple. We consider the security condition for $W = X_0 \cup X_{r-1} \cup Y_0 \cup Y_{t-1}$:

$$\begin{aligned}2|X_0| + 2|X_{r-1}| + 2|Y_0| + 2|Y_{t-1}| - 4 &\geq |N[W] \cap S| \\ &\geq |N[W] - S| \\ &\geq |X_0| + 2 + |X_{r-1}| + 2 + |Y_0| + 2 + \\ &\quad + |Y_{t-1}| + 2 + 4,\end{aligned}$$

which leads to

$$|X_0| + |X_{r-1}| + |Y_0| + |Y_{t-1}| \geq 16. \tag{1}$$

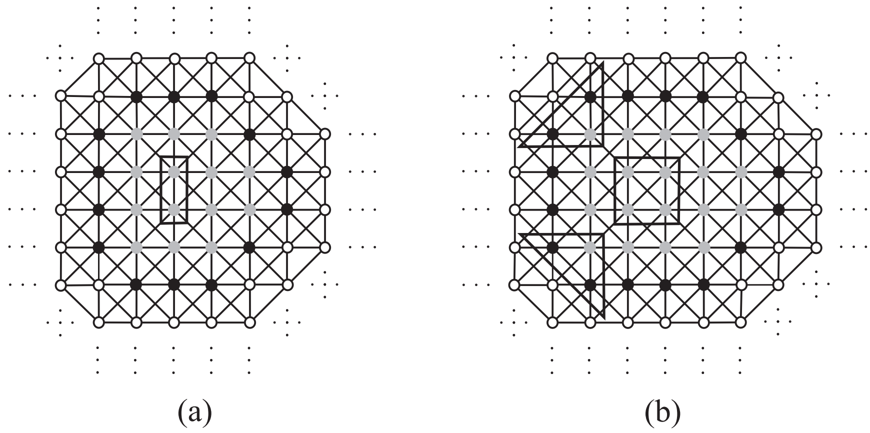


Figure 10: Possible borders of S with exactly two simple corner-like sets which are lying over either X_0 or X_{r-1}.

However, since $r \geq 6$, there exists a set X_{r-3}. We observe that $N[X_2] \cap X_0 = \emptyset$ and $N[X_{r-3}] \cap X_{r-1} = \emptyset$ according to the construction of the graph. Moreover, the structure of S implies that $|N[Y_0] \cap X_2| = 2$, $|N[Y_0] \cap X_{r-3}| = 2$, $|N[Y_{t-1}] \cap X_2| = 2$ and $|N[Y_{t-1}] \cap X_{r-3}| = 2$. Hence,

$$|N[W] \cap S| \leq |X_0| + |X_1| + \ldots + |X_{r-1}| - (|X_0| - 2) - (|X_{r-1}| - 2)$$

and

$$|N[W] - S| \geq |X_0| + 4 + |X_{r-1}| + 4 + 2(r-2) + 4.$$

Since $|N[W] \cap S| \geq |N[W] - S|$,

$$|X_0| + |X_1| + \ldots + |X_{r-1}| \geq 2(|X_0| + |X_{r-1}|) + 2r + 4. \tag{2}$$

Analogously, since $t \geq 6$, there exists a set Y_{t-3}. From the construction of the graph it follows that $N[Y_2] \cap Y_0 = \emptyset$ and $N[Y_{t-3}] \cap Y_{t-1} = \emptyset$. Also, $|N[X_0] \cap Y_2| = 2$, $|N[X_0] \cap Y_{t-3}| = 2$, $|N[X_{r-1}] \cap Y_2| = 2$ and $|N[X_{r-1}] \cap Y_{t-3}| = 2$. Thus,

$$|Y_0| + |Y_1| + \ldots + |Y_{t-1}| \geq 2(|Y_0| + |Y_{t-1}|) + 2t + 4. \tag{3}$$

Since $|S| = |X_0| + \ldots + |X_{r-1}| = |Y_0| + \ldots + |Y_{t-1}|$, we get from (1), (2) and

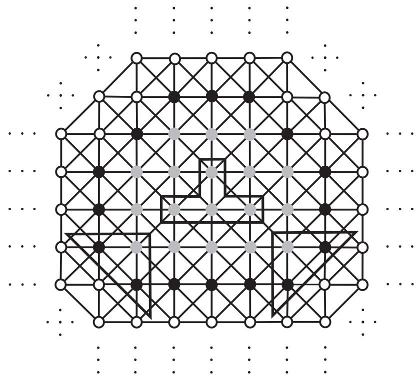


Figure 11: Possible border of S with exactly two simple corner-like sets which are lying over either Y_0 or Y_{t-1}.

(3)

$$\begin{aligned}2|S| &\geq 2(|X_0| + |X_{r-1}|) + 2r + 4 + 2(|Y_0| + |Y_{t-1}|) + 2t + 4 \quad \text{(from (2) and (3))}\\ &= 2(|X_0| + |X_{r-1}| + |Y_0| + |Y_{t-1}|) + 2r + 2t + 8\\ &\geq 32 + 2r + 2t + 8 \qquad \text{(from (1))}\\ &= 2r + 2t + 40,\end{aligned}$$

which leads to $|S| \geq r + t + 20$. Moreover, since $r, t \geq 6$, it follows that $|S| \geq 32$, which is a contradiction with the first assumption.

Therefore, if S is a secure set of minimum cardinality in $C_m \boxtimes C_n$, then $|S| \geq \min\{2m, 2n, 32\}$ and the proof is completed. $\square$

Proposition 15. *For any integers $m, n \geq 2$,*

$$s(P_m \boxtimes C_n) \geq \min\{2m, n, 16\}.$$

Proof. The proof is relatively analogous to the proof of Proposition 14. Suppose that $s(P_m \boxtimes C_n) < \min\{2m, n, 16\}$ and let S be a $s(P_m \boxtimes C_n)$-set. We consider the sets X_α, $\alpha \in \{0, \ldots, m-1\}$ and Y_β, $\beta \in \{0, \ldots, n-1\}$.

First, if $X_0 = \emptyset$ and $X_{m-1} = \emptyset$, then S has the same structure like in the case of a torus (Proposition 14) and $|S| \geq \min\{2m, 2n, 32\}$, which is a contradiction. Thus, $X_0 \neq \emptyset$ or $X_{m-1} \neq \emptyset$. We assume for instance that $X_0 \neq \emptyset$. Since $s(P_m \boxtimes C_n) < \min\{2m, n, 16\}$, for every $\alpha \in \{0, \ldots, m-1\}$, $|X_\alpha| < n$.

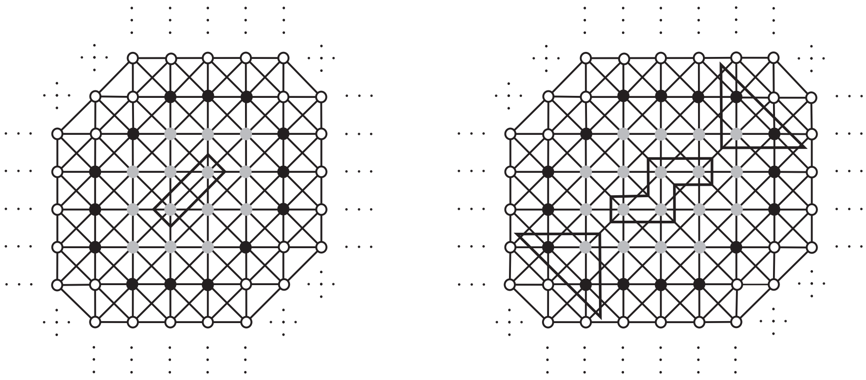


Figure 12: Possible borders of S with exactly two simple corner-like sets such that only one of them is lying over X_0.

Suppose there exists a set Y_k such that $|Y_k| = m$. If $S = Y_k$, then $|N[S] \cap S| = 1/2|N[S] - S|$, a contradiction. So, $Y_k \subsetneq S$. Since S is connected, $Y_{k-1} \neq \emptyset$ or $Y_{k+1} \neq \emptyset$. We assume that $Y_{k-1} \neq \emptyset$. Moreover $|Y_{k-1}| < m$ because $s(P_m \boxtimes C_n) < 2m$. Thus, $|N[S] \cap S| < 2m \leq |N[S] - S|$, which is a contradiction. Hence, $|S \cap Y_{k-1}| \geq m$ and $|S| \geq 2m$.

As a consequence, for every $\alpha \in \{0, \ldots, m-1\}$ and $\beta \in \{0, \ldots, n-1\}$ we have $|X_\alpha| < n$ and $|Y_\beta| < m$. Hence, like in the proof of Proposition 14, S is a subset of some rectangle $R(0, r-1, g, h)$. We assume without loss of generality that $g = 0$ and $h = t-1$. By Lemma 13, $3 \leq r \leq 4$ and $6 \leq t \leq 8$. If $r = 4$ or $t = 8$, then by Lemma 8, $|N[S] - S| \geq 16$. Thus, $|S| \geq 16$, which is a contradiction. Therefore, $r = 3$ and $6 \leq t \leq 7$.

From now on the proof is very similar to the one of Proposition 14, so we explain only its crucial steps. We consider only the corner-like sets $C(r-1, l')$, $l' \in \{0, t-1\}$. For these sets one can formulate claims (i)-(iv) similarly as in the previous proof. Moreover, we know that $r = 3$ and $6 \leq t \leq 7$.

Case 1: No corner-like set of S is simple. Hence, S can have one of the borders illustrated in Figure 14. However, in both cases the security condition is not satisfied for $\mathcal{BD}$.

Case 2: Only one corner-like set of S is simple. All possible borders of S are drawn in Figure 15. From Figure 15 (a), it follows that $|N[\mathcal{BD}] \cap S| \leq 13$ and $|N[\mathcal{BD}] - S| \geq 14$, a contradiction. Moreover, in Figure 15 (b) we have $|N[\mathcal{BD}] \cap S| = |N[\mathcal{BD}] - S| = 15$ only if all gray vertices that have a

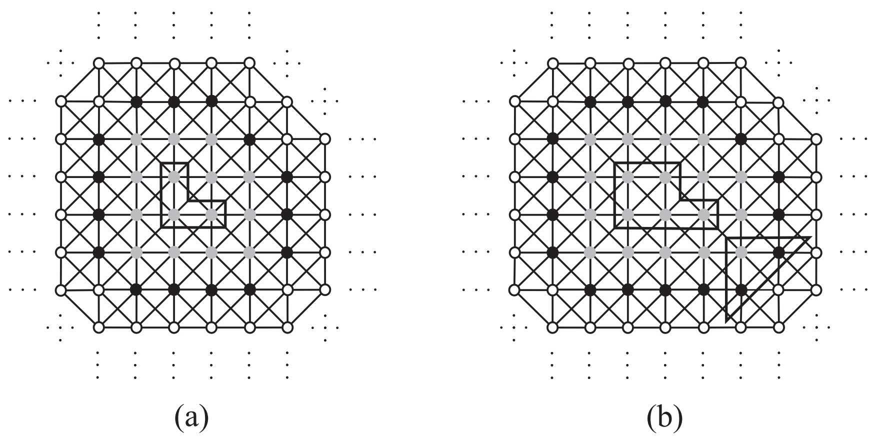


Figure 13: Possible borders of S with exactly three simple corner-like sets.

neighbour in $\mathcal{BD}$ belong to S. Otherwise, S is not secure. Since $|S| \leq 15$, the two vertices in the rectangle, which are not adjacent to any vertex in $\mathcal{BD}$, do not belong to S. Therefore, for the set T of vertices inside the drawn triangle we have $|N[T] \cap S| = 7$ and $|N[T] - S| = 8$. Hence, the set S is not secure, a contradiction.

Case 3: Both corner-like sets of S are simple. We consider the security condition for $W = X_{r-1} \cup Y_0 \cup Y_{t-1}$. We have

$$|N[W] \cap S| \leq 2|X_{r-1}| + 2|Y_0| + 2|Y_{t-1}|$$

and

$$|N[W] - S| \geq |X_{r-1}| + 2 + |Y_0| + 1 + |Y_{t-1}| + 1 + 2.$$

From the security of S and the inequalities above it follows that

$$|X_{r-1}| + |Y_0| + |Y_{t-1}| \geq 8. \tag{4}$$

Since $r \geq 3$, there exists X_{r-3}. We observe that $N[X_{r-3}] \cap X_{r-1} = \emptyset$ and $|N[Y_0] \cap X_{r-3}| = 2$, which follows from the construction of the graph. Moreover, by the structure of S, $|N[Y_{t-1}] \cap X_{r-3}| = 2$. Hence,

$$|N[W] \cap S| \leq |X_0| + |X_1| + \ldots + |X_{r-1}| - (|X_{r-1}| - 2)$$

and

$$|N[W] - S| \geq |X_{r-1}| + 4 + 2(r-1) + 2.$$

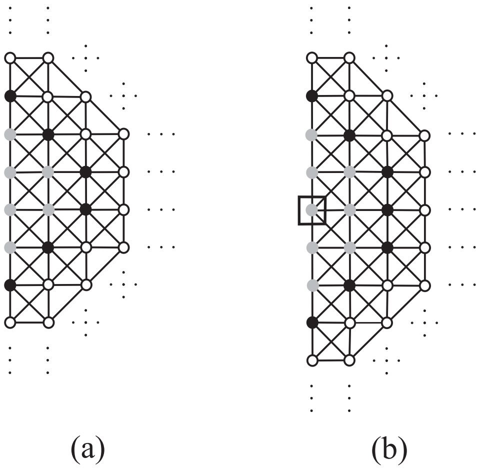


Figure 14: Possible borders of S with no simple corner-like sets.

Because $|N[W] \cap S| \geq |N[W] - S|$,

$$|X_0| + |X_1| + \ldots + |X_{r-1}| \geq 2|X_{r-1}| + 2r + 2. \tag{5}$$

Analogously, if $t \geq 6$, we obtain

$$|Y_0| + |Y_1| + \ldots + |Y_{t-1}| \geq 2(|Y_0| + |Y_{t-1}|) + t + 2. \tag{6}$$

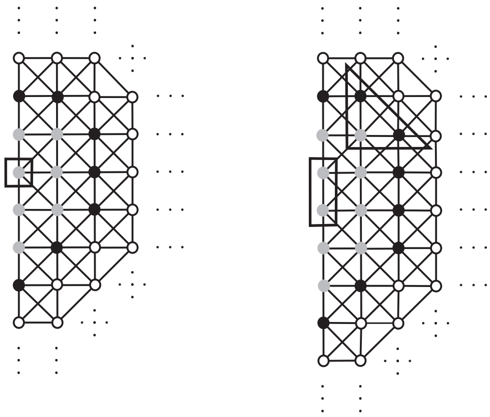
Since $|S| = |X_0| + \ldots + |X_{r-1}| = |Y_0| + \ldots + |Y_{t-1}|$, we get from (4), (5) and (6)

$$\begin{aligned} 2|S| &\geq 2|X_{r-1}| + 2r + 2 + 2(|Y_0| + |Y_{t-1}|) + t + 2 \quad \text{(from (5) and (6))} \\ &= 2(|X_{r-1}| + |Y_0| + |Y_{t-1}|) + 2r + t + 4 \\ &\geq 16 + 2r + t + 4 \qquad \text{(from (4))} \\ &= 2r + t + 20, \end{aligned}$$

which leads to $|S| \geq r + t/2 + 10$. Since $r \geq 3$ and $t \geq 6$, we get $|S| \geq 16$, and this is a contradiction with our first assumption. This observation finishes the proof. □

Proposition 16. *For any integers $m, n \geq 2$,*

$$s(P_m \boxtimes P_n) \geq \min\{m, n, 8\}.$$



(a) (b)

Figure 15: Possible borders of S with exactly one simple corner-like set.

Proof. Suppose to the contrary that $s(P_m \boxtimes P_n) < \min\{m, n, 8\}$. Let S be a $s(P_m \boxtimes P_n)$-set. Consider the sets X_α and Y_β, where $\alpha \in \{0, \ldots, m-1\}$ and $\beta \in \{0, \ldots, n-1\}$. If $X_0 = \emptyset$, $X_{m-1} = \emptyset$, $Y_0 = \emptyset$ and $Y_{n-1} = \emptyset$, then similarly as in the proof of Proposition 14 we can show that $|S| \geq \min\{2m, 2n, 32\}$, which is a contradiction. Moreover, if three of the sets X_0, X_{m-1}, Y_0 and Y_{n-1} are empty, then we can use the same arguments as in the proof of Proposition 15 to prove that $|S| \geq \min\{2m, n, 16\}$, a contradiction. Thus, at least two of the sets X_0, X_{m-1}, Y_0 and Y_{n-1} are not empty. Since S is connected, if at least three of the sets X_0, X_{m-1}, Y_0 and Y_{n-1} are not empty, then we obtain that $|S| \geq m$ or $|S| \geq n$. Similarly, if the sets (X_0 and X_{m-1}) are not empty or the sets (Y_0 and Y_{n-1}) are not empty, then $|S| \geq m$ or $|S| \geq n$, respectively.

Now suppose that X_0 and Y_{n-1} are not empty. Consider the rectangle $R(0, f, g, n-1)$. If $f+1 = r = 2$, then, since $n-1-g+1 = n-g = t < n-1$, there exist two vertices $x, y \in (X_0 \cup X_1) \cap (Y_{n-1} \cup Y_{n-2})$ (these vertices are marked with squares in Figure 16 (a) and (b)), such that either ($|N[\{x, y\}] \cap S| \leq 4$ and $|N[\{x, y\}] - S| \geq 5$) (See Figure 16 (a)) or ($|N[\{x, y\}] \cap S| \leq 5$ and $|N[\{x, y\}] - S| \geq 6$) (See Figure 16 (b)), which is a contradiction. It follows that $r \geq 3$. Analogously we deduce that $t \geq 3$. Thus, we have that the rectangle $R(0, f, g, n-1)$ has at least nine vertices. One can check that no subset of less than eight vertices of $R(0, f, g, n-1)$ is a secure set. Therefore $|S| \geq 8$, and this contradiction finishes the proof. $\square$

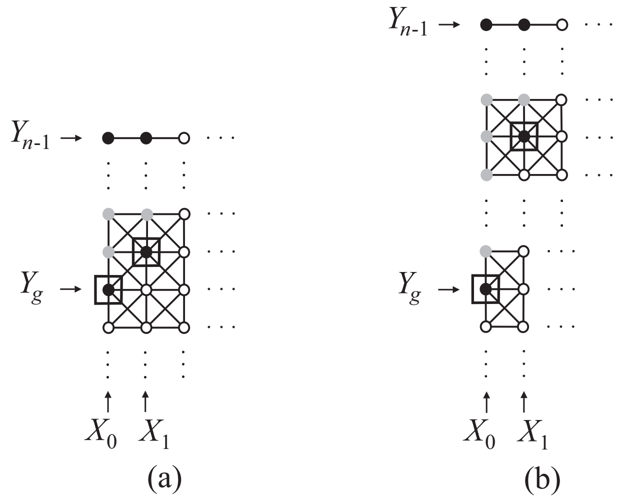


Figure 16: Possible cases of corner-like sets of $P_m \boxtimes P_n$.

As a conclusion of the sections above, the equalities of Theorem 2 for the security number of the strong product graphs $P_m \boxtimes P_n$, $P_m \boxtimes C_n$ and $C_m \boxtimes C_n$ are proved.

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