S-packing colorings of distance graphs with distance sets of cardinality 2

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Abstract

For a non-decreasing sequence $S = (s_1, s_2, \ldots)$ of positive integers, a partition of the vertex set of a graph G into subsets $X_1, \ldots, X_\ell$, such that vertices in X_i are pairwise at distance greater than s_i for every $i \in \{1, \ldots, \ell\}$, is called an S-packing ℓ-coloring of G. The minimum ℓ for which G admits an S-packing ℓ-coloring is called the S-packing chromatic number of G, denoted by $\chi_S(G)$. In this paper, we consider S-packing colorings of distance graphs $G(\mathbb{Z}, \{k,t\})$, where k and t are positive integers, which are the graphs whose vertex set is $\mathbb{Z}$, and two vertices $x, y \in \mathbb{Z}$ are adjacent whenever $|x - y| \in \{k, t\}$. We complement partial results from two earlier papers, thus determining all values of $\chi_S(G(\mathbb{Z}, \{k,t\}))$ when S is any sequence with $s_i \leq 2$ for all i. In particular, if $S = (1, 1, 2, 2, \ldots)$, then the S-packing chromatic number is 2 if $k + t$ is even, and 4 otherwise, while if $S = (1, 2, 2, \ldots)$, then the S-packing chromatic number is 5, unless $\{k, t\} = \{2, 3\}$ when it is 6; when $S = (2, 2, 2, \ldots)$, the corresponding formula is more complex.

Key words: S-packing coloring, S-packing chromatic number, distance graph, distance coloring.

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1 Introduction

Given a graph G and a non-decreasing sequence $S = (s_1, s_2, \ldots)$ of positive integers, the mapping $f : V(G) \to [\ell] = \{1, \ldots, \ell\}$ is an *S-packing ℓ-coloring* of G if for any distinct vertices $u, v \in V(G)$ with $f(u) = f(v) = i$, $i \in \{1, \ldots, \ell\}$, the distance between u and v in G is greater than s_i. The smallest ℓ such that G has an S-packing ℓ-coloring is the *S-packing chromatic number* of G, denoted by $\chi_S(G)$. This concept was introduced by Goddard, Hedetniemi, Hedetniemi, Harris, and Rall [13], and was studied in a number of papers; see the recent survey [3] and the references therein. The main focus of the seminal paper and a number of subsequent papers was on the specific sequence $S = (n)_{n\geq 1}$ in which positive integers appear in the natural order, where the resulting graph invariant is simply called the packing chromatic number [4]. Goddard and Xu [14] started consideration of various non-decreasing sequences S, and a number of authors followed them. Arguably the most interesting sequences are those that involve only integers 1 and 2, since they are in a sense between standard coloring, where S is the constant sequence of 1s, and 2-distance coloring, where S is the constant sequence of 2s (note that a 2-distance coloring is equivalent to a coloring of the square of a graph and it has been intensively studied in the last decades [17]). In particular, S-packing colorings of subcubic graphs were investigated with respect to such sequences S [5,12,18]. Roughly a decade ago, Ekstein et al. [10] and Togni [19] initiated the study of S-packing colorings in integer distance graphs, which we present next. Their study was motivated by a series of papers on proper vertex coloring of distance graphs, see e.g. [6–9] and references therein.

Given a set $D = \{d_1, \ldots, d_h\}$, $h \geq 1$, of positive integers, the *(integer) distance graph*, $G(\mathbb{Z}, D)$, is the infinite graph with $\mathbb{Z}$ as the vertex set, while vertices x and y are adjacent if $|x - y| \in D$. That is, two vertices/integers are adjacent in the graph if their distance in $\mathbb{Z}$ is one of the integers in $\{d_1, \ldots, d_h\}$. We will simplify the notation, and instead of $G(\mathbb{Z}, D)$ write $G(D)$, and for distance sets with two integers we will write $D = \{k, t\}$, and always assume that $k < t$; thus the corresponding distance graph will be written as $G(k, t)$. The packing chromatic numbers of distance graphs $G(k, t)$ were investigated in [11]. In addition, the S-packing colorings of distance graphs $G(k, t)$ where $k \in \{1, 2\}$ and t is arbitrary were studied in [2, 16]. Concerning the sequences S which involve only integers that are not greater than 2, exact values for $\chi_S(G(1, t))$, where $t \geq 2$, were determined in [1, 16], while the values $\chi_S(G(2, t))$, where $t \geq 3$, were established in [2]. Hence, for this type of sequences S, the S-packing chromatic numbers of $G(k, t)$ were left open when $k \geq 3$, and the main goal of this paper is to establish these remaining values.

In the next section, we establish the notation and give some preliminary observations. In particular, we present two main tools that are used in the proofs. First, we present a representation of the graph $G(k, t)$ in the so-called shifted grid. Second, we introduce color patterns and shift sequences, which enable us a relatively brief presentation of colorings. In Section 3, we follow with proving the main results, which are the values of $\chi_S(G(k, t))$, for all $3 \leq k < t$, and for all possible sequences S involving only integers 1 and 2. In Section 4, we give an overview of results on S-packing colorings of the graph $G(k, t)$ by combining the results from [2, 16] with new results from this paper.

2 Notation and preliminaries

When presenting sequences S, we will often use i^p, where i and p are positive integers, as a shortened notation of the (sub)sequence $(i,\ldots,i)$, where i appears p times. For instance, $(1^2,2^5)$ stands for the (sub)sequence $(1,1,2,2,2,2,2)$. We may also write i^∞ which coincides with the infinite (sub)sequence $(i,i,i,...)$. In the case of distinct integers in the sequence S, the integer with the power to infinity is the largest among the integers in S. For instance, $(1^2,2^\infty)$ presents the sequence with two integers 1 and all other integers 2.

Note that $G(k,t)$ is connected if and only if $\gcd(k,t)=1$. Thus, when determining the S-packing chromatic numbers, we restrict to graphs $G(k,t)$ such that k and t are coprime integers. Note that if $\gcd(k,t)=g$, then $G(k,t)$ consists of connected components all of which are graphs $G(\frac{k}{g},\frac{t}{g})$, implying that $\chi_S(G(k,t))=\chi_S(G(\frac{k}{g},\frac{t}{g}))$.

During our study we make use of the following presentation. Notably, a connected distance graph $G(k,t)$ can be represented by the square lattice $\{0,1,\ldots,t\}\times\mathbb{Z}$ with vertices given by *points*, which are ordered pairs $(i,j), i\in\{0,1,\ldots,t\}, j\in\mathbb{Z}$, such that vertex/point (i,j) of the grid is a representative of the integer $j\cdot t+i\cdot k$ from $G(k,t)$. These representatives are unique with the exception of the points whose first coordinate equals to 0 or t, since a point $(0,j)$ on the grid represents the same integer of $G(k,t)$ as the point (t,j'), where $j'=j-k$. See Fig. 1, where ordered pairs in red present points on the grid, while integers in black present the corresponding integers from $\mathbb{Z}$.

Furthermore, let *column* i denote the set of vertices $B_i=\{(i,j): j\in\mathbb{Z}\}$, where $0\le i\le t$. As mentioned earlier, integers from $V(G(k,t))$ of the form $jt, j\in\mathbb{Z}$, are represented twice on the grid, notably by a vertex in the column 0 and a vertex in column t. That is, jt is represented by vertex $(0,j)$ as well as vertex $(t,j-k)$. For instance, points $(0,0)$ and $(t,-k)$ represent $0\in V(G(k,t))$.

2.1 Color patterns

In this paper, we will often present an S-packing coloring c by using periodic patterns applied on columns B_i, where $i\in\{0,\ldots,t\}$. A periodic pattern of length $d\ge 2$ is a sequence of colors $[c_1,\ldots,c_d]$ (denoted by square brackets), where the colors $c_n, n\in[d]$, are not necessarily pairwise distinct. These colors are given to consecutive vertices within one column and the pattern is applied downwards. That is, if the coloring c is using a pattern $P=[c_1,\ldots,c_d]$ in the column B_x such that $c(x,y)=c_1$ for some y, then $c(x,y-n)=c_{n+1}$ for each $n\in[d-1]$. This pattern is then periodically copied upwards and downwards to cover all the vertices of B_x. Thus, $c(x,y+1)=c_d$, $c(x,y+2)=c_{d-1}$ and so on.

In order for c to be an S-packing coloring of $G(k,t)$, patterns must be often shifted in consecutive columns. We define the notion of *shift sequence* $(p_i)_{i=0}^{t-1}$ where $p_i\in\mathbb{N}_0$. To describe it, we also need the concept of *reference point* (i,j) in B_i, which is a unique point in each column. Without loss of generality we may declare the reference point in column B_0 to be $(0,0)$. The integer $p_i\ge 0$ represents the value by which the reference point in B_{i+1} shifts downwards with respect to the reference point in B_i. That is, if

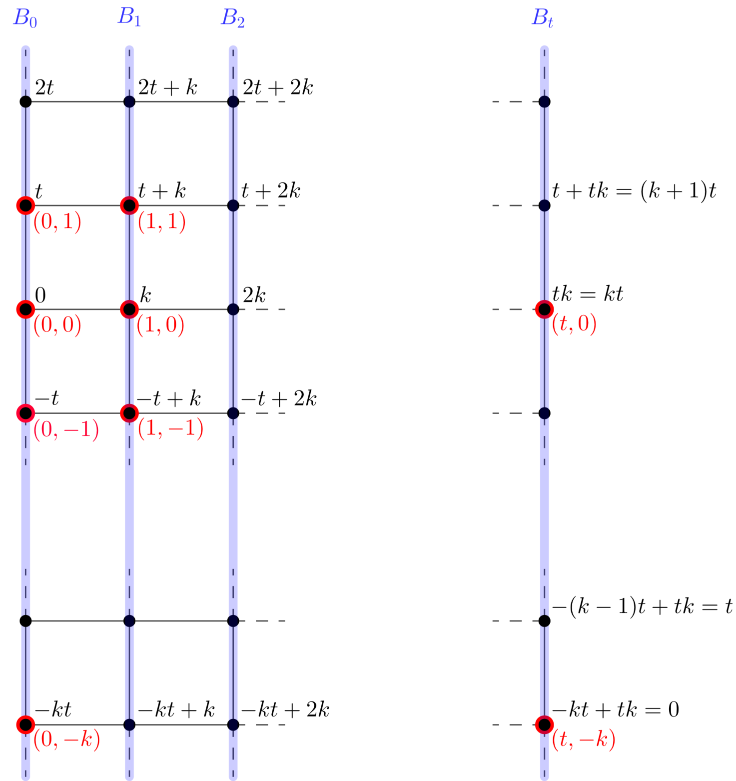

Figure 1: Representation of the distance graph $G(k,t)$ in the square grid.

the reference point in B_i is (i,j), for some $j \in \mathbb{Z}$, then $(i+1, j-p_i)$ is the reference point in column B_{i+1}. (Note that $p_i = 0$ means that there is no shift.) To make the elements of the shift sequence correspond to the application of periodic patterns to the columns B_i, we will always assume that the reference point of each column B_i always receives the first color of the corresponding pattern applied to this column. That is, if a coloring c is using the pattern $P = [c_1, \ldots, c_d]$ in a column B_x with the reference point (x,j) then $c(x,j) = c_1$. Thus, the pattern P used together with the reference point (x,j) completely determines the coloring of points in B_x.

When shifting the patterns, there are two cases to consider: either columns B_i and B_{i+1} are using the same pattern or they are using different patterns. First, suppose that a coloring c uses the same pattern $[c_1, \ldots, c_d]$ in columns B_i and B_{i+1}. In this case the integer p_i defines the value by which the color c_1 in B_{i+1} is shifted with respect to the color c_1 in B_i. In other words, if $c(i,j) = c_1$ for some $j \in \mathbb{Z}$, then $c(i+1, j-p_i) = c_1$. This implies that $p_i \neq 0$ (the pattern must be shifted) since the adjacent vertices must not receive the same color. Next, let c assign a pattern $[a_1, \ldots, a_{d_1}]$ to the column B_i and a pattern $[b_1, \ldots, b_{d_2}]$ to the column B_{i+1}. In this case, the integer p_i defines the value by which the color b_1 given to the reference point of B_{i+1} is shifted with respect to the color a_1 given to the reference point of B_i. This means that if $c(i, j) = a_1$, where (i, j) is the reference point in B_i, then $c(i+1, j-p_i) = b_1$ (note that $(i+1, j-p_i)$ is the reference point in B_{i+1}).

Since the points $(0, j)$ represent the same vertices as points $(t, j-k)$, the relation $c(0, j) = c(t, j-k)$ must hold for each $j \in \mathbb{Z}$. Thus, if c is an S-packing coloring of $G(k, t)$, then columns B_0 and B_t must use the same periodic pattern and $\sum_{i=0}^{t-1} p_i \equiv k \pmod{d}$, where d is the length of the pattern used on B_0. Note that it is sufficient to consider only shifts $p_i < d_{\max}$ where $d_{\max}$ denotes the length of the longest pattern.

In this paper, all colorings given by periodic patterns and shift sequences will either use one or two different patterns. If the second option occurs, it is necessary to specify which column receives which pattern. Let c be an S-packing coloring of $G(k, t)$ given by the shift sequence $(p_i)_{i=0}^{t-1}$ and two periodic patterns $P_1 = [a_1, \ldots, a_{d_1}]$ and $P_2 = [b_1, \ldots, b_{d_2}]$ such that pattern P_2 is used for example on columns B_1 and B_3, and the rest of the columns obtain pattern P_1. We will abbreviate such description as:

$$[a_1, \ldots, a_{d_1}]_{p_0}[b_1, \ldots, b_{d_2}]_{p_1}[a_1, \ldots, a_{d_1}]_{p_2}[b_1, \ldots, b_{d_2}]_{p_3}[a_1, \ldots, a_{d_1}]_{p_{4\to t-1}}[a_1, \ldots, a_{d_1}],$$

provided that the shift sequence has $p_4 = \ldots = p_{t-1}$.

For a better understanding of the concepts, we next provide a more specific example. Let c be an (1^6)-coloring of $G(k, t)$ given by $P_1 = [1, 2, 3]$, $P_2 = [4, 5, 6]$ and the shift sequence $(0, 1, 1, 0, 1^{t-4})$ where the notation 1^{t-4} stands for the (sub)sequence $(1, \ldots, 1)$, where 1 appears $(t-4)$ times. Again, the pattern P_2 is applied in columns B_1 and B_3, while the rest of the columns use the pattern P_1. Thus, the (1^6)-coloring c is given by:

$$[1, 2, 3]_{p_0=0}[4, 5, 6]_{p_1=1}[1, 2, 3]_{p_2=1}[4, 5, 6]_{p_3=0}[1, 2, 3]_{p_{4\to t-1}=1}[1, 2, 3].$$

Figure 2 demonstrates how the reference point (marked in red) is shifted with respect to the given shift sequence and how the patterns are applied.

To verify that c is an S-packing coloring of $G(k, t)$ we will use equivalent conditions, which can be derived from the above notation. Notably, c is an S-packing coloring of $G(k, t)$ if and only if every two vertices with the same color are at sufficient distance, columns B_0 and B_t obtain the same pattern and $\sum_{i=0}^{t-1} p_i \equiv k \pmod{d}$, where d is the length of the pattern in B_0. The first among these three conditions requires the following verification. Note that for every two points (x, y) and (u, v) their distance in $G(k, t)$ equals $\min\{|x-u|+|y-v|, x+(t-u)+|(y-k)-v|\}$. Hence, if $c(x, y) = i = c(u, v)$, then

$$\min\{|x-u| + |y-v|, x + (t-u) + |(y-k) - v|\} > s_i.$$

In particular, if $s_i = 1$, then it is sufficient to verify that two vertices with color i are not adjacent in the grid. Similarly, if $s_i = 2$, then two vertices with color i must not be adjacent in the grid, must not have a common neighbor in the grid, and $c(1, j) = i$ implies $c(t-1, j-k) \neq i$ and vice versa.

3 S-packing colorings of graphs $G(k, t)$

In this section, we present values of $\chi_S(G(k, t))$, where $3 \leq k < t$ are positive integers, for all possible infinite sequences S whose elements are in $\{1, 2\}$.

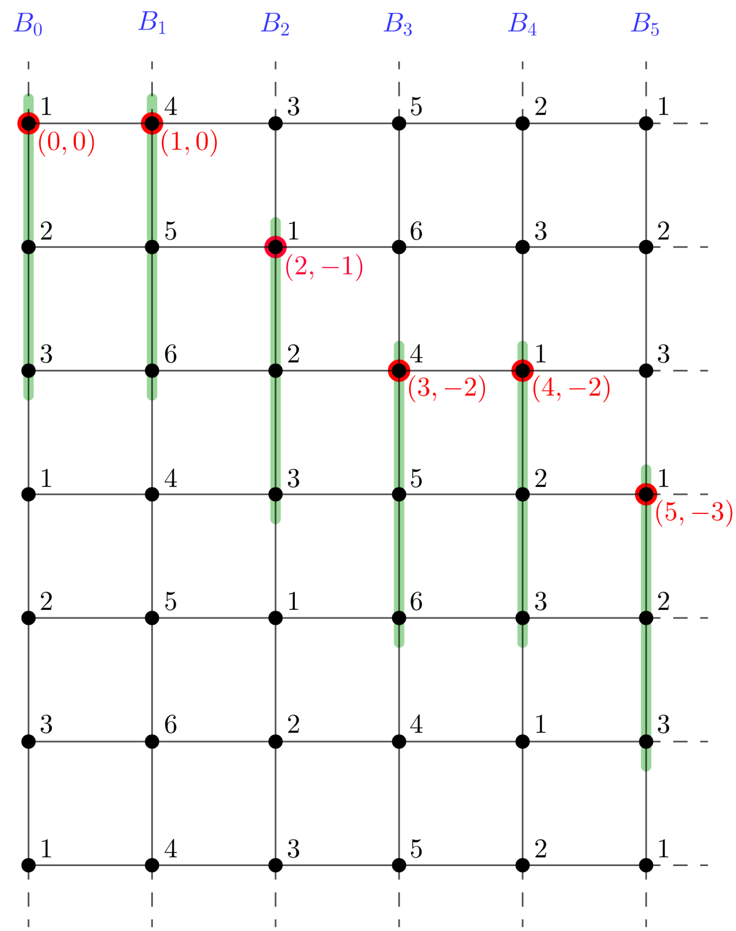

Figure 2: (1^6)-coloring of $G(k,t)$ given by patterns $[1,2,3]$ and $[4,5,6]$, and the shift sequence $(0,1,1,0,1^{t-4})$.

For completeness of our study of S-packing colorings of graphs $G(k,t)$, we first recall the known results about the standard chromatic number. That is, we consider the sequence $S=(1^\infty)$. Concerning the chromatic number of distance graphs $G(D)$, Walther proved the general bound $\chi(G(D)) \le |D|+1$; see [1, 20]. Since $|D|=2$ in our case, we infer that $\chi(G(k,t)) \in \{2,3\}$ depending on whether $G(k,t)$ is bipartite or not. Notably, for $3 \le k < t$, we derive

$$\chi(G(k,t)) = \begin{cases} 2; & k+t \textit{ even}, \\ 3; & k+t \textit{ odd}. \end{cases}$$

Therefore, whenever a sequence S contains three 1s, the above result may be applied. In the next subsection, we deal with the three remaining subcases of sequences S depending on the number of 1s.

3.1 $S = (1, 1, 2, 2, 2 \dots)$

Theorem 3.1 *If $G(k,t)$ is the distance graph, where k, t are coprime positive integers such that $3 \le k < t$, and $S = (1, 1, 2^\infty)$, then*

$$\chi_S(G(k,t)) = \begin{cases} 2; & k+t \text{ even}, \\ 4; & k+t \text{ odd}. \end{cases}$$

Proof. If $k+t$ is even, $G(k,t)$ is bipartite, hence $\chi_S(G(k,t)) = 2$. In the rest of the proof, we assume $k+t$ is odd, which implies $\chi_S(G(k,t)) \ge 3$.

Suppose that $\chi_S(G(k,t)) = 3$, and let $c : V(G(k,t)) \to [3]$ be a $(1,1,2)$-coloring of $G(k,t)$. Suppose that there is a vertex $(x,y) \in V(G(k,t))$ in column B_x, where $x \notin \{0,t\}$, with $c(x,y) = 3$. For this choice of x, the neighborhood of (x,y) is the set $N((x,y)) = \{(x,y+1),(x-1,y),(x+1,y),(x,y-1)\}$.

Note that all vertices in $N((x,y))$ have to receive either color 1 or 2, since every two vertices with color 3 must be at the distance at least 3. Let $c(x,y+1) = a$, where $a \in \{1,2\}$, and let $\{a,b\} = \{1,2\}$. Hence, we derive the following chain of implications:

$$\begin{aligned} &c(x,y+1) = a \Rightarrow c(x+1,y+1) = b \Rightarrow c(x+1,y) = a \Rightarrow c(x+1,y-1) = b \\ &\Rightarrow c(x,y-1) = a \Rightarrow c(x-1,y-1) = b \Rightarrow c(x-1,y) = a \Rightarrow c(x-1,y+1) = b. \end{aligned}$$

We infer that, for every vertex (x,y) with $c(x,y) = 3$, all neighbors receive the same color. (If $x \in \{0,t\}$, we get the same conclusion either by checking the neighborhood of (x,y) as above, or by noting that columns B_0 and B_t represent the same yet shifted column by which the coloring can be reassigned so that the corresponding point with color 3 is in one of columns between 1 and $t-1$.) Hence, the coloring $c' : V(G(k,t)) \to [2]$ obtained from c by recoloring every vertex (i,j) with $c(i,j) = 3$ by using the color in $\{1,2\}$ which does not appear in its neighborhood with respect to coloring c is a $(1,1)$-coloring of $G(k,t)$. Thus, $G(k,t)$ is bipartite, which is a contradiction with the assumption that $k+t$ is odd. We derive that $\chi_S(G(k,t)) \ge 4$.

For the proof of the upper bound, we present a $(1,1,2,2)$-coloring c given by the shift sequence $(p_i)_{i=0}^{t-1} = (0,2,0,1^{t-3})$ and two periodic patterns $[1,2]$ and $[3,4,2,1]$ such that the pattern $[3,4,2,1]$ is used on columns B_1 and B_2, and rest of the columns obtain pattern $[1,2]$. That is, c is defined by:

$$[1,2]_{p_0=0}[3,4,2,1]_{p_1=2}[3,4,2,1]_{p_2=0}[1,2]_{p_{3\to t-1}=1}[1,2].$$

Note that c is a $(1,1,2,2)$-coloring of $G(k,t)$ if and only if three conditions hold: every two vertices with the same color are at sufficient distance, columns B_0 and B_t must obtain the same pattern and $\sum_{i=0}^{t-1} p_i \equiv k \pmod 2$. The matrix below demonstrates the presented coloring c of $G(k,t)$ in the first five columns (note that bold integers indicate the location of reference points):

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & \mathbf{3} & 2 & 1 & 2 \\
2 & 4 & 1 & 2 & 1 \\
1 & 2 & \mathbf{3} & \mathbf{1} & 2 \\
2 & 1 & 4 & 2 & \mathbf{1} \\
1 & 3 & 2 & 1 & 2 \\
2 & 4 & 1 & 2 & 1 \\
1 & 2 & 3 & 1 & 2 \\
2 & 1 & 4 & 2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It is easy to verify that all vertices with the same color are at sufficient distance and due to $t \geq 4$, columns B_0 and B_t always use the same pattern $[1, 2]$. Thus, the first and the second condition hold. From the following sum we obtain:

$$\sum_{i=0}^{t-1} p_i = 2 + (t-3) = t - 1 \equiv \begin{cases} 0 \pmod 2; & t \textit{ odd}, \\ 1 \pmod 2; & t \textit{ even}. \end{cases}$$

Since k and t are of opposite parity, the condition $\sum_{i=0}^{t-1} p_i \equiv k \pmod 2$ also holds.

Therefore, $\chi_S(G(k,t)) = 4$ for $k+t$ odd. □

3.2 $S = (1, 2, 2, 2, \dots)$

Theorem 3.2 *If $G(k,t)$ is the distance graph, where k,t are coprime positive integers such that $3 \leq k < t$, and $S = (1, 2^\infty)$, then $\chi_S(G(k,t)) = 5$.*

Proof. Due to the representation of $G(k,t)$ as the (shifted) square grid $\{0,1,\dots,t\} \times \mathbb{Z}$, when determining the lower bound of $\chi_S(G(k,t))$, we can use results known for the infinite grid $\mathbb{Z}^2$. Goddard and Xu [15] proved that $\chi_S(\mathbb{Z}^2) = 5$, hence $\chi_S(G(k,t)) \geq 5$. (Alternatively, the same conclusion is derived when observing a vertex in $G(k,t)$ receiving color 1, and noting that its neighbors must receive pairwise distinct colors from $\{2,\dots,5\}$.)

Next, we determine $(1, 2^4)$-colorings of $G(k,t)$ with respect to the values k and t.

1. Let $t \geq 12$. In this case, we present six different $(1, 2^4)$-colorings c_n, where $n \in \{0, 1, 2, 3, 4, 5\}$, which are then applied with respect to k and t. The first $(1, 2^4)$-coloring c_0 is given by just one periodic pattern $A = [1, 2, 3, 1, 4, 5]$ and the shift sequence $\mathbf{q_0} = (2^t)$, while the rest of c_n given by the shift sequences $\mathbf{q_n}, n \in \{1, 2, 3, 4, 5\}$, also use another pattern $B = [4, 1, 5, 2, 1, 3]$ in addition to

pattern A. More precisely,

$$\begin{aligned}
\mathbf{q_1} &= (0,5,2^{t-2}),\\
\mathbf{q_2} &= ((0,5)^2,2^{t-4}),\\
\mathbf{q_3} &= ((0,5)^3,2^{t-6}),\\
\mathbf{q_4} &= ((0,5)^4,2^{t-8}),\\
\mathbf{q_5} &= ((0,5)^5,2^{t-10}),
\end{aligned}$$

and

$$\begin{aligned}
&c_1 : A_{p_0=0}B_{p_1=5}A_{p_{2\to t-1}=2}A,\\
&c_2 : A_{p_0=0}B_{p_1=5}A_{p_2=0}B_{p_3=5}A_{p_{4\to t-1}=2}A,\\
&c_3 : A_{p_0=0}B_{p_1=5}A_{p_2=0}B_{p_3=5}A_{p_4=0}B_{p_5=5}A_{p_{6\to t-1}=2}A,\\
&c_4 : A_{p_0=0}B_{p_1=5}A_{p_2=0}B_{p_3=5}A_{p_4=0}B_{p_5=5}A_{p_6=0}B_{p_7=5}A_{p_{8\to t-1}=2}A,\\
&c_5 : A_{p_0=0}B_{p_1=5}A_{p_2=0}B_{p_3=5}A_{p_4=0}B_{p_5=5}A_{p_6=0}B_{p_7=5}A_{p_8=0}B_{p_9=5}A_{p_{10\to t-1}=2}A.
\end{aligned}$$

Recall that c_n is a $(1,2^4)$-coloring of $G(k,t)$ if and only if three conditions hold: every two vertices with the same color are at sufficient distance, columns B_0 and B_t obtain the same pattern, and $\sum_{i=0}^{t-1} p_i \equiv k \pmod 6$. From the definition of colorings c_n we immediately see that both columns B_0 and B_t obtain the pattern A, hence the second condition holds.

To verify the first condition, we consider the possible pattern layouts in three consecutive columns, B_i, B_{i+1}, B_{i+2}, using the patterns A and B. Since B_0 and B_t represent the same, yet shifted column, the case when $i = t-1$ is interpreted as B_{t-1}, B_t, B_1. We have five possibilities: AAA, BAA, ABA, AAB and BAB. Note that when AAA is used, in all presented sequences the two shifts are $p_i = 2 = p_{i+1}$, and it is easy to verify that the points with the same colors are at sufficient distances. This correspondence will be presented as:

$$AAA \longleftrightarrow 2,2.$$

Similarly, we have the following correspondences between pattern layouts and shifts, which can be derived from the definitions of sequences $\mathbf{q_n}$:

$$\begin{aligned}
BAA &\longleftrightarrow 5,2,\\
ABA &\longleftrightarrow 0,5,\\
AAB &\longleftrightarrow 2,0,\\
BAB &\longleftrightarrow 5,0.
\end{aligned}$$

(Note that AAB appears when considering the columns B_{t-1}, B_t and B_1.) The corresponding matrices for each of the five cases are placed below.

$$
\begin{array}{ccc}
A & A & A \\
\vdots & \vdots & \vdots \\
\mathbf{1} & 4 & 3 \\
2 & 5 & 1 \\
3 & \mathbf{1} & 4 \\
1 & 2 & 5 \\
4 & 3 & \mathbf{1} \\
5 & 1 & 2 \\
1 & 4 & 3 \\
2 & 5 & 1 \\
\vdots & \vdots & \vdots
\end{array}
\qquad
\begin{array}{ccc}
A & A & B \\
\vdots & \vdots & \vdots \\
\mathbf{1} & 4 & 1 \\
2 & 5 & 3 \\
3 & \mathbf{1} & \mathbf{4} \\
1 & 2 & 1 \\
4 & 3 & 5 \\
5 & 1 & 2 \\
1 & 4 & 1 \\
2 & 5 & 3 \\
\vdots & \vdots & \vdots
\end{array}
\qquad
\begin{array}{ccc}
A & B & A \\
\vdots & \vdots & \vdots \\
\mathbf{1} & \mathbf{4} & 2 \\
2 & 1 & 3 \\
3 & 5 & 1 \\
1 & 2 & 4 \\
4 & 1 & 5 \\
5 & 3 & \mathbf{1} \\
1 & 4 & 2 \\
2 & 1 & 3 \\
\vdots & \vdots & \vdots
\end{array}
\qquad
\begin{array}{ccc}
B & A & A \\
\vdots & \vdots & \vdots \\
\mathbf{4} & 2 & 5 \\
1 & 3 & 1 \\
5 & 1 & 2 \\
2 & 4 & 3 \\
1 & 5 & 1 \\
3 & \mathbf{1} & 4 \\
4 & 2 & 5 \\
1 & 3 & \mathbf{1} \\
\vdots & \vdots & \vdots
\end{array}
\qquad
\begin{array}{ccc}
B & A & B \\
\vdots & \vdots & \vdots \\
\mathbf{4} & 2 & 1 \\
1 & 3 & 5 \\
5 & 1 & 2 \\
2 & 4 & 1 \\
1 & 5 & 3 \\
3 & \mathbf{1} & \mathbf{4} \\
4 & 2 & 1 \\
1 & 3 & 5 \\
\vdots & \vdots & \vdots
\end{array}
$$

Again, in all cases it is easy to verify that the points with the same colors are at sufficient distances. In this way, the first condition is also verified.

What remains is to use the third condition $\sum_{i=0}^{t-1} p_i \equiv k \pmod 6$ to determine which sequence $\mathbf{q_n}$ is suitable for $G(k,t)$ with respect to values k, t. Let $t \equiv \ell \pmod 6$ where $\ell \in \{0,1,2,3,4,5\}$. For each $\mathbf{q_n}$ we calculate the value of $\sum_{i=0}^{t-1} p_i$:

$$\mathbf{q_n} : \sum_{i=0}^{t-1} p_i = 5n + 2(t-2n) = 2t + n.$$

Due to appropriate numbering of sequences we obtain $\sum_{i=0}^{t-1} p_i \equiv (2\ell + n) \pmod 6$ for each $\mathbf{q_n}$. In order to determine which c_n gives us the $(1, 2^4)$-coloring of $G(k,t)$ for fixed values k, t, let $k \equiv m \pmod 6$ where $m \in \{0,1,2,3,4,5\}$. Using the condition $\sum_{i=0}^{t-1} p_i \equiv k \pmod 6$ we derive:

$$(2\ell + n) \pmod 6 = m \implies n = (m - 2\ell) \pmod 6.$$

Thus, if $k \equiv m \pmod 6$ and $t \equiv \ell \pmod 6$ then the $(1, 2^4)$-coloring of $G(k,t)$ is given by c_n using the sequence $\mathbf{q_n}$ such that $n = (m - 2\ell) \pmod 6$.

2. Let $t = 11$, hence $k \in \{3,4,5,6,7,8,9,10\}$. For $G(3,11)$ and $G(8,11)$ we present a $(1, 2^4)$-coloring given by the periodic pattern $[1,2,3,4,5]$ and the shift sequence $(p_i)_{i=0}^{10} = (3^{11})$. The matrix below demonstrates the presented $(1, 2^4)$-coloring of

these graphs:

$$
\begin{array}{cccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & 3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 & 3 \\
2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 \\
3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 & 3 & 5 \\
4 & \mathbf{1} & 3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 \\
5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 \\
1 & 3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 & 3 \\
2 & 4 & \mathbf{1} & 3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 \\
3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 & 3 & 5 \\
4 & 1 & 3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

It is easy to verify that all vertices with the same color are at a sufficient distance and the condition $\sum_{i=0}^{10} p_i \equiv 3 \pmod 5$ holds for both graphs.

For $G(4,11)$, $G(5,11)$, $G(6,11)$, $G(7,11)$, $G(9,11)$ and $G(10,11)$ we use $(1,2^4)$-colorings c_0, c_1, c_2, c_3, c_5 and c_0 from case 1, respectively.

3. Let $t = 10$, hence $k \in \{3,7,9\}$. For both $G(3,10)$ and $G(9,10)$ we use $(1,2^4)$-coloring c_1 from case 1.

   For $G(7,10)$ we present a $(1,2^4)$-coloring given by two periodic patterns $C = [1,2,1,3,1,4,1,5]$ and $D = [4,3,5,4,2,5,3,2]$, and the shift sequence $(p_i)_{i=0}^{9} = ((0,1)^3, 3^4)$ such that:

$$C_{p_0=0} D_{p_1=1} C_{p_2=0} D_{p_3=1} C_{p_4=0} D_{p_5=1} C_{p_{6\to 9}=3} C.$$

   The matrix below demonstrates the presented $(1,2^4)$-coloring of this graph:

$$
\begin{array}{ccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & \mathbf{4} & 5 & 2 & 1 & 3 & 4 & 1 & 5 & 1 & 2 \\
2 & 3 & \mathbf{1} & \mathbf{4} & 5 & 2 & 1 & 3 & 1 & 4 & 1 \\
1 & 5 & 2 & 3 & \mathbf{1} & \mathbf{4} & 5 & 1 & 2 & 1 & 3 \\
3 & 4 & 1 & 5 & 2 & 3 & \mathbf{1} & 4 & 1 & 5 & 1 \\
1 & 2 & 3 & 4 & 1 & 5 & 2 & 1 & 3 & 1 & 4 \\
4 & 5 & 1 & 2 & 3 & 4 & 1 & 5 & 1 & 2 & 1 \\
1 & 3 & 4 & 5 & 1 & 2 & 3 & \mathbf{1} & 4 & 1 & 5 \\
5 & 2 & 1 & 3 & 4 & 5 & 1 & 2 & 1 & 3 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

   It is easy to verify that all vertices with the same color are at a sufficient distance and the condition $\sum_{i=0}^{9} p_i \equiv 7 \pmod 8$ holds.

4. Let $t = 9$, hence $k \in \{4,5,7,8\}$. For $G(4,9)$ we present a $(1,2^4)$-coloring given by two periodic patterns $C = [1,2,1,3,1,4,1,5]$ and $D = [4,3,5,4,2,5,3,2]$, and

the shift sequence $(p_i)_{i=0}^{8} = ((0,1)^3, 3^3)$ such that:

$$C_{p_0=0}D_{p_1=1}C_{p_2=0}D_{p_3=1}C_{p_4=0}D_{p_5=1}C_{p_{6\to 8}=3}C.$$

The matrix below demonstrates the presented $(1,2^4)$-coloring of this graph:

$$\begin{array}{cccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & \mathbf{4} & 5 & 2 & 1 & 3 & 4 & 1 & 5 & 1 \\
2 & 3 & \mathbf{1} & \mathbf{4} & 5 & 2 & 1 & 3 & 1 & 4 \\
1 & 5 & 2 & 3 & \mathbf{1} & \mathbf{4} & 5 & 1 & 2 & 1 \\
3 & 4 & 1 & 5 & 2 & 3 & \mathbf{1} & 4 & 1 & 5 \\
1 & 2 & 3 & 4 & 1 & 5 & 2 & 1 & 3 & 1 \\
4 & 5 & 1 & 2 & 3 & 4 & 1 & 5 & 1 & 2 \\
1 & 3 & 4 & 5 & 1 & 2 & 3 & \mathbf{1} & 4 & 1 \\
5 & 2 & 1 & 3 & 4 & 5 & 1 & 2 & 1 & 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It is easy to verify that all vertices with the same color are at a sufficient distance and the condition $\sum_{i=0}^{8} p_i \equiv 4 \pmod 8$ holds.

For $G(5,9)$ we present a $(1,2^4)$-coloring given by the periodic pattern $C = [1,2,1,3,1,4,1,5]$ and the shift sequence $(p_i)_{i=0}^{8} = (5^9)$. Again, the matrix below demonstrates the presented $(1,2^4)$-coloring of this graph:

$$\begin{array}{cccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & 3 & 1 & 2 & 1 & 5 & 1 & 4 & 1 & 3 \\
2 & 1 & 5 & 1 & 4 & 1 & 3 & 1 & 2 & 1 \\
1 & 4 & 1 & 3 & 1 & 2 & 1 & 5 & 1 & 4 \\
3 & 1 & 2 & 1 & 5 & 1 & 4 & 1 & 3 & 1 \\
1 & 5 & 1 & 4 & 1 & 3 & 1 & 2 & 1 & 5 \\
4 & \mathbf{1} & 3 & 1 & 2 & 1 & 5 & 1 & 4 & 1 \\
1 & 2 & 1 & 5 & 1 & 4 & 1 & 3 & 1 & 2 \\
5 & 1 & 4 & 1 & 3 & 1 & 2 & 1 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It is easy to verify that all vertices with the same color are at a sufficient distance and the condition $\sum_{i=0}^{8} p_i \equiv 5 \pmod 8$ holds.

For $G(7,9)$ and $G(8,9)$ we use $(1,2^4)$-colorings c_1 and c_2 from case 1, respectively.

5. Let $t = 8$, hence $k \in \{3,5,7\}$. For $G(3,8)$ we present a $(1,2^4)$-coloring given by two periodic patterns $C = [1,2,1,3,1,4,1,5]$ and $D = [4,3,5,4,2,5,3,2]$, and the shift sequence $(p_i)_{i=0}^{7} = (0,1,3^6)$ such that:

$$C_{p_0=0}D_{p_1=1}C_{p_{2\to 7}=3}C.$$

The matrix below demonstrates the presented $(1, 2^4)$-coloring of this graph:

$$\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & \mathbf{4} & 5 & 1 & 2 & 1 & 3 & 1 & 4 \\
2 & 3 & \mathbf{1} & 4 & 1 & 5 & 1 & 2 & 1 \\
1 & 5 & 2 & 1 & 3 & 1 & 4 & 1 & 5 \\
3 & 4 & 1 & 5 & 1 & 2 & 1 & 3 & 1 \\
1 & 2 & 3 & \mathbf{1} & 4 & 1 & 5 & 1 & 2 \\
4 & 5 & 1 & 2 & 1 & 3 & 1 & 4 & 1 \\
1 & 3 & 4 & 1 & 5 & 1 & 2 & 1 & 3 \\
5 & 2 & 1 & 3 & \mathbf{1} & 4 & 1 & 5 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It is easy to verify that all vertices with the same color are at a sufficient distance and the condition $\sum_{i=0}^{7} p_i \equiv 3 \pmod 8$ holds.

For $G(5, 8)$ and $G(7, 8)$ we use $(1, 2^4)$-colorings c_1 and c_3 from case 1, respectively.

6. Let $t = 7$, hence $k \in \{3, 4, 5, 6\}$. For $G(3, 7)$, $G(4, 7)$ and $G(5, 7)$ we use $(1, 2^4)$-colorings c_1, c_2 and c_3 from case 1, respectively.

   For $G(6, 7)$ we present a $(1, 2^4)$-coloring given by the periodic pattern $[1, 2, 3, 4, 5]$ and the shift sequence $(p_i)_{i=0}^{6} = (3^7)$. The matrix below demonstrates the presented $(1, 2^4)$-coloring of this graph:

$$\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & 3 & 5 & 2 & 4 & 1 & 3 & 5 \\
2 & 4 & 1 & 3 & 5 & 2 & 4 & 1 \\
3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 \\
4 & \mathbf{1} & 3 & 5 & 2 & 4 & 1 & 3 \\
5 & 2 & 4 & 1 & 3 & 5 & 2 & 4 \\
1 & 3 & 5 & 2 & 4 & 1 & 3 & 5 \\
2 & 4 & \mathbf{1} & 3 & 5 & 2 & 4 & 1 \\
3 & 5 & 2 & 4 & 1 & 3 & 5 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

   It is easy to verify that all vertices with the same color are at a sufficient distance and $\sum_{i=0}^{6} p_i \equiv 1 \pmod 5$, hence the condition $\sum_{i=0}^{6} p_i \equiv k \pmod 5$ holds.

7. Let $t = 6$, hence $k = 5$. For $G(5, 6)$ we present an $(1, 2^4)$-coloring given by two periodic patterns $C = [1, 2, 1, 3, 1, 4, 1, 5]$ and $D = [4, 3, 5, 4, 2, 5, 3, 2]$, and the shift sequence $(p_i)_{i=0}^{5} = (0, 1, 3^4)$ such that:

$$C_{p_0=0} D_{p_1=1} C_{p_{2\to5}=3} C.$$

The matrix below demonstrates the presented $(1, 2^4)$-coloring of this graph.

$$\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & \mathbf{4} & 5 & 1 & 2 & 1 & 3 \\
2 & 3 & \mathbf{1} & 4 & 1 & 5 & 1 \\
1 & 5 & 2 & 1 & 3 & 1 & 4 \\
3 & 4 & 1 & 5 & 1 & 2 & 1 \\
1 & 2 & 3 & \mathbf{1} & 4 & 1 & 5 \\
4 & 5 & 1 & 2 & 1 & 3 & 1 \\
1 & 3 & 4 & 1 & 5 & 1 & 2 \\
5 & 2 & 1 & 3 & \mathbf{1} & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It is easy to verify that all vertices with the same color are at a sufficient distance and the condition $\sum_{i=0}^{4} p_i \equiv 5 \pmod 8$ holds.

8. Let $t = 5$, hence $k \in \{3, 4\}$. For $G(3, 5)$ we present an $(1, 2^4)$-coloring given by the periodic pattern $[(1, 2, 1, 3)^2, (1, 4, 1, 5)^2]$ and the shift sequence $(p_i)_{i=0}^{4} = (7^5)$. The matrix below demonstrates the presented $(1, 2^4)$-coloring of this graph.

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & 4 & 1 & 5 & 1 & 4 \\
2 & 1 & 3 & 1 & 2 & 1 \\
1 & 5 & 1 & 4 & 1 & 5 \\
3 & 1 & 2 & 1 & 3 & 1 \\
1 & 4 & 1 & 5 & 1 & 2 \\
2 & 1 & 3 & 1 & 4 & 1 \\
1 & 5 & 1 & 2 & 1 & 3 \\
3 & \mathbf{1} & 4 & 1 & 5 & 1 \\
1 & 2 & 1 & 3 & 1 & 2 \\
4 & 1 & 5 & 1 & 4 & 1 \\
1 & 3 & 1 & 2 & 1 & 3 \\
5 & 1 & 4 & 1 & 5 & 1 \\
1 & 2 & 1 & 3 & 1 & 4 \\
4 & 1 & 5 & 1 & 2 & 1 \\
1 & 3 & \mathbf{1} & 4 & 1 & 5 \\
5 & 1 & 2 & 1 & 3 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

It is easy to verify that all vertices with the same color are at a sufficient distance and the condition $\sum_{i=0}^{4} p_i \equiv 3 \pmod{16}$ holds.

For $G(4, 5)$ we use coloring c_0 from case 1.

9. Let $t = 4$, hence $k = 3$. For $G(3, 4)$ we use coloring c_1 from case 1.

After verifying all the possible cases for k and t, we conclude that $\chi_S(G(k,t)) = 5$, and the proof is complete. $\square$

3.3 $S = (2, 2, 2, \dots)$

Lemma 3.3 *If $G(k,t)$ is the distance graph, where k, t are coprime positive integers such that $3 \le k < t$, and $S = (2^\infty)$, then $\chi_S(G(k,t)) \le 6$.*

Proof. We determine (2^6)-colorings of $G(k,t)$ with respect to the values of k, t.

1. Let $t \ge 6$. In this case, we present six different (2^6)-colorings c_n, where $n \in \{0,1,2,3,4,5\}$, which we apply with respect to k and t. Each c_n is given by the periodic pattern $[1,2,3,4,5,6]$ and a shift sequence $\mathbf{q_n} = (p_i)_{i=0}^{t-1}$ such that:

$$\begin{aligned}
\mathbf{q_0} &= (2^t),\\
\mathbf{q_1} &= (3, 2^{t-1}),\\
\mathbf{q_2} &= (3, 2, 3, 2^{t-3}),\\
\mathbf{q_3} &= (3, 2, 3, 2, 3, 2^{t-5}),\\
\mathbf{q_4} &= (3, 4, 3, 2^{t-3}),\\
\mathbf{q_5} &= (3, 4, 3, 2, 3, 2^{t-5}).
\end{aligned}$$

Since the colorings are given by the unique pattern, c_n is a (2^6)-coloring of $G(k,t)$ if and only if every two vertices with the same color are at the distance at least 3 and $\sum_{i=0}^{t-1} p_i \equiv k \pmod 6$. To verify the first condition, we observe a coloring $c : V(\mathbb{Z}^2) \to [6]$ of the infinite grid $\mathbb{Z}^2$ using the pattern $[1,2,3,4,5,6]$. Since c is (2^6)-coloring, the following must hold for each $i, j \in \mathbb{Z}$:

$$\begin{aligned}
c(i,j) &\ne c(i+1, j),\\
c(i,j) &\ne c(i+1, j \pm 1),\\
c(i,j) &\ne c(i+2, j).
\end{aligned}$$

Thus, the periodic pattern $[1,2,3,4,5,6]$ in column B_{i+1} can be shifted by $2, 3$ or 4 with respect to the column B_i. In addition, the pattern in column B_{i+2} can be shifted by $1, 2, 3, 4$ or 5 with respect to the column B_i. Thus, for the graph $G(k,t)$ we obtain $p_i \in \{2,3,4\}$, and

$$\begin{aligned}
p_i = 2 &\Rightarrow p_{i+1} \ne 4,\\
p_i = 3 &\Rightarrow p_{i+1} \ne 3,\\
p_i = 4 &\Rightarrow p_{i+1} \ne 2,
\end{aligned}$$

holds for all $i \in \{0, \dots, t-2\}$, and, furthermore,

$$\begin{aligned}
p_0 = 2 &\Rightarrow p_{t-1} \ne 4,\\
p_0 = 3 &\Rightarrow p_{t-1} \ne 3,\\
p_0 = 4 &\Rightarrow p_{t-1} \ne 2,
\end{aligned}$$

because B_0 and B_t represents the same yet shifted column.

Next, we use the second condition $\sum_{i=0}^{t-1} p_i \equiv k \pmod 6$ to determine which sequence $\mathbf{q_n}$ is suitable for $G(k,t)$ with respect to values k, t. Let $t \equiv \ell \pmod 6$ where $\ell \in \{0,1,2,3,4,5\}$. For each $\mathbf{q_n}$ we calculate the value of $\sum_{i=0}^{t-1} p_i$:

$\mathbf{q_0} : \sum_{i=0}^{t-1} p_i = 2t \equiv 2\ell \pmod 6$,

$\mathbf{q_1} : \sum_{i=0}^{t-1} p_i = 3 + 2(t-1) = 2t + 1 \equiv (2\ell + 1) \pmod 6$,

$\mathbf{q_2} : \sum_{i=0}^{t-1} p_i = 3 + 2 + 3 + 2(t-3) = 2t + 2 \equiv (2\ell + 2) \pmod 6$,

$\mathbf{q_3} : \sum_{i=0}^{t-1} p_i = 3 + 2 + 3 + 2 + 3 + 2(t-5) = 2t + 3 \equiv (2\ell + 3) \pmod 6$,

$\mathbf{q_4} : \sum_{i=0}^{t-1} p_i = 3 + 4 + 3 + 2(t-3) = 2t + 4 \equiv (2\ell + 4) \pmod 6$,

$\mathbf{q_5} : \sum_{i=0}^{t-1} p_i = 3 + 4 + 3 + 2 + 3 + 2(t-5) = 2t + 5 \equiv (2\ell + 5) \pmod 6$.

Thus, for $\mathbf{q_n}$ we obtain $\sum_{i=0}^{t-1} p_i \equiv (2\ell + n) \pmod 6$. In order to determine which c_n gives us the (2^6)-coloring of $G(k,t)$ for fixed values k, t, let $k \equiv m \pmod 6$ where $m \in \{0,1,2,3,4,5\}$. From the condition $\sum_{i=0}^{t-1} p_i \equiv k \pmod 6$ we obtain:

$$(2\ell + n) \pmod 6 = m \implies n = (m - 2\ell) \pmod 6,$$

hence if $k \equiv m \pmod 6$ and $t \equiv \ell \pmod 6$ then the (2^6)-coloring of $G(k,t)$ is given by c_n using the sequence $\mathbf{q_n}$ such that $n = (m - 2\ell) \pmod 6$.

2. Let $t = 5$, hence $k \in \{3, 4\}$. For $G(3,5)$ we present a (2^6)-coloring given by the periodic pattern $[1,2,3,4,5,1,6,3,2,5,4,6]$ and the shift sequence $(p_i)_{i=0}^{4} = (3^5)$. The matrix below demonstrates the presented (2^6)-coloring of this graph.

$$\begin{matrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{1} & 5 & 6 & 4 & 1 & 5 \\
2 & 4 & 3 & 5 & 2 & 4 \\
3 & 6 & 2 & 1 & 3 & 6 \\
4 & \mathbf{1} & 5 & 6 & 4 & 1 \\
5 & 2 & 4 & 3 & 5 & 2 \\
1 & 3 & 6 & 2 & 1 & 3 \\
6 & 4 & \mathbf{1} & 5 & 6 & 4 \\
3 & 5 & 2 & 4 & 3 & 5 \\
2 & 1 & 3 & 6 & 2 & 1 \\
5 & 6 & 4 & \mathbf{1} & 5 & 6 \\
4 & 3 & 5 & 2 & 4 & 3 \\
6 & 2 & 1 & 3 & 6 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{matrix}$$

It is easy to verify that all vertices with the same color are at a sufficient distance and the condition $\sum_{i=0}^{4} p_i \equiv 3 \pmod{12}$ holds.

For $G(4,5)$ we use the coloring c_0 from case 1.

3. Let $t = 4$, hence $k = 3$. For $G(3,4)$ we use the coloring c_1 from case 1.

By considering all cases of k and t, we obtain $\chi_S(G(k,t)) \leq 6$, as desired. $\square$

Theorem 3.4 *If $G(k,t)$ is the distance graph, where k,t are coprime positive integers such that $3 \leq k < t$, and $S = (2^\infty)$, then*

$$\chi_S(G(k,t)) = \begin{cases} 5; & (t \equiv 1,4 \pmod 5 \text{ and } k \equiv 2,3 \pmod 5) \\ & \text{or } (t \equiv 2,3 \pmod 5 \text{ and } k \equiv 1,4 \pmod 5)), \\ & \\ 6; & \text{otherwise.} \end{cases}$$

Proof. Consider the infinite grid $\mathbb{Z}^2$. Goddard and Xu [15] proved that $\chi_S(\mathbb{Z}^2) = 5$, hence $\chi_S(G(k,t)) \geq 5$. Let $c : V(\mathbb{Z}^2) \to [5]$ be a (2^5)-coloring of $\mathbb{Z}^2$. Consider a vertex $(x,y) \in V(\mathbb{Z}^2)$ with its neighborhood $N((x,y)) = \{(x,y+1),(x-1,y),(x+1,y),(x,y-1)\}$. Each of these 5 vertices must receive a different color since they are at the distance at most 2. Without loss of generality, let

$$c(x,y+1) = 1,\ c(x-1,y) = 2,\ c(x,y) = 3,\ c(x+1,y) = 4,\ c(x,y-1) = 5.$$

Thus, the vertex $(x+1,y+1)$ can obtain either color 2 or 5 with respect to the coloring c. We consider both options.

1. Let $c(x+1,y+1) = 2$. Due to the assignment of colors to vertices in $N([x,y])$ and $(x+1,y+1)$, we derive the following chain of implications:

$$\begin{aligned} & c(x+1,y+1) = 2 \Rightarrow c(x+1,y-1) = 1 \Rightarrow c(x-1,y-1) = 4 \\ & \Rightarrow c(x,y-2) = 2 \Rightarrow c(x+1,y-2) = 3 \Rightarrow c(x-1,y-2) = 1 \\ & \Rightarrow c(x,y-3) = 4 \Rightarrow c(x+1,y-3) = 5 \Rightarrow c(x-1,y-3) = 3 \\ & \Rightarrow c(x,y-4) = 1 \Rightarrow \dots \end{aligned}$$

   Thus, the coloring c forces for each column B_i the periodic pattern $[1,3,5,2,4]$. Moreover, $c(i,j) = c(i+1,j-2)$ for all $i,j \in \mathbb{Z}$.

2. Let $c(x+1,y+1) = 5$. Similarly:

$$\begin{aligned} & c(x+1,y+1) = 5 \Rightarrow c(x-1,y+1) = 4 \Rightarrow c(x-1,y-1) = 1 \\ & \Rightarrow c(x+1,y-1) = 2 \Rightarrow c(x,y-2) = 4 \Rightarrow c(x-1,y-2) = 3 \\ & \Rightarrow c(x+1,y-2) = 1 \Rightarrow c(x,y-3) = 2 \Rightarrow c(x-1,y-3) = 5 \\ & \Rightarrow c(x+1,y-3) = 3 \Rightarrow c(x,y-4) = 1 \Rightarrow \dots \end{aligned}$$

   In this case, the coloring c forces for each column B_i the periodic pattern $[1,3,5,4,2]$ and $c(i,j) = c(i+1,j-3)$ for all $i,j \in \mathbb{Z}$.

We have shown that (up to permutation of colors) there exist only two 2^5-packing colorings of the square lattice $\mathbb{Z}^2$. We now apply these two colorings to $G(k,t)$.

1. Consider the coloring given by the periodic pattern $[1,3,5,2,4]$ with relation $c(i,j) = c(i+1, j-2)$. Hence, this coloring enforces the constant shift sequence $(p_i)_{i=0}^{t-1}$ where $p_i = 2$ for all $i \in \{0,\dots,t-1\}$. Note that if c is a (2^5)-coloring of $G(k,t)$, then $\sum_{i=0}^{t-1} p_i \equiv k \pmod 5$. What remains is to determine for which values of k, t this condition holds.

   Let $t \equiv \ell \pmod 5$ where $\ell \in \{0,1,2,3,4\}$. From the following sum we derive:

$$\sum_{i=0}^{t-1} p_i = \sum_{i=0}^{t-1} 2 = 2t \equiv 2\ell \pmod 5,$$

   hence $k \equiv 2\ell \pmod 5$. Note that for $\ell = 0$ we obtain $t, k \equiv 0 \pmod 5$ which is a contradiction with the proposition $\gcd(k,t) = 1$ and thus it is sufficient to consider $\ell \in \{1,2,3,4\}$.

2. Consider the coloring given by the periodic pattern $[1,3,5,4,2]$ with relation $c(i,j) = c(i+1, j-3)$, which is equivalent to the constant shift sequence $(p_i)_{i=0}^{t-1}$ where $p_i = 3$ for all $i \in \{0,\dots,t-1\}$. We again determine for which k, t the condition $\sum_{i=0}^{t-1} p_i \equiv k \pmod 5$ holds.

   Let $t \equiv \ell \pmod 5$ where $\ell \in \{0,1,2,3,4\}$. Similarly to the previous case:

$$\sum_{i=0}^{t-1} p_i = \sum_{i=0}^{t-1} 3 = 3t \equiv 3\ell \pmod 5,$$

   hence $k \equiv 3\ell \pmod 5$. For $\ell = 0$ we obtain $t, k \equiv 0 \pmod 5$, contradicting $\gcd(k,t) = 1$ again, thus we consider $\ell \in \{1,2,3,4\}$.

By calculating the values of k, t for the given ℓ, we obtain that $\chi_S(G(k,t)) = 5$ if and only if either ($t \equiv 1,4 \pmod 5$ and $k \equiv 2,3 \pmod 5$) or ($t \equiv 2,3 \pmod 5$ and $k \equiv 1,4 \pmod 5$). From Lemma 3.3 we derive $\chi_S(G(k,t)) = 6$ for the remaining values of k, t. □

4 Concluding remarks

These results presented in this paper complement previously known results from [1, 2, 16, 20] on S-packing colorings of connected distance graphs $G(k,t)$, which satisfy the property that each integer in a sequence S belongs to $\{1,2\}$. Note that such S-packing colorings are classical colorings ($S = (1^\infty)$), 2-distance colorings ($S = (2^\infty)$) and S-packing colorings which lie between these two.

The amalgamation of the results presented herein with those established earlier gives us the S-packing chromatic numbers of all connected distance graphs $G(k,t)$, where $k \geq 1$ and $t > k$ are coprime. With respect to the sequence S we summarize all of these results as follows.

1. $S = (1^\infty)$.

$$\chi(G(k,t)) = \begin{cases} 2; & k+t \text{ even}, \\ 3; & k+t \text{ odd}. \end{cases}$$

2. $S = (1,1,2^\infty)$.

$$\chi_S(G(k,t)) = \begin{cases} 2; & k+t \text{ even}, \\ 4; & k+t \text{ odd}. \end{cases}$$

3. $S = (1,2^\infty)$.

$$\chi_S(G(k,t)) = \begin{cases} 5; & k \neq 2 \text{ or } t \neq 3, \\ 6; & \text{otherwise}. \end{cases}$$

4. $S = (2^\infty)$.

$$\chi_2(G(k,t)) = \begin{cases} 5; & (t \equiv 1,4 \pmod 5 \text{ and } k \equiv 2,3 \pmod 5) \\ & \quad \text{or } (t \equiv 2,3 \pmod 5 \text{ and } k \equiv 1,4 \pmod 5), \\ & \\ 7; & k = 2 \text{ and } t = 3, \\ 6; & \text{otherwise}. \end{cases}$$

As for other sequences S that contain elements greater than 2, investigations of $\chi_S(G(k,t))$ are far from complete. For the sequence $S = (d^\infty)$, $d \geq 3$, lower and upper bounds are known for $\chi_S(G(k,t))$ [2], which in some cases culminate to exact results. For instance, if $t \geq 5$ is an odd integer and $d \geq t-3$, then $\chi_d(G(k,t)) = 1 + t \cdot \left(d - \frac{t-3}{2}\right)$. Similarly, lower and upper bounds are known for the sequence $S = (1,2,3,\ldots)$ which corresponds to the standard packing coloring. This results focus on the distance graph $G(1,t)$ [10,19]. Additionally, exact results exist for certain sporadic sequences S, which came into fruition while studying the S-packing chromatic numbers of $G(k,t)$ for sequences that contain only elements from $\{1,2\}$. The S-packing coloring of $G(1,t)$ provided in [16] partitions the color classes in such a way that the vertices of some color classes are farther apart than they need to be. As a consequence, this gives results for the sequences S with larger elements.

There remains ample scope for research concerning the S-packing chromatic number of distance graphs $G(k,t)$. In addition, for distance graphs $G(D)$, where $|D| \geq 3$, only a few results on their S-packing colorings are known.

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