On the security number of the Cartesian product of graphs

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Abstract

A *secure set* of a graph is, intuitively, a set that can refute any attack from the neighborhood to its subsets. Formally, it is defined as a set $S \subseteq V(G)$ such that $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$. Although finding a minimum secure set is a computationally intractable problem, the minimum size of secure sets, called the *security number*, is studied for some specific graphs. Especially, determining the security number of the Cartesian product of graphs is one of the developed directions in this area. In this paper, we present an upper bound on the security number of the Cartesian product of general graphs, which is tight for some sparse graphs. We then determine the security number of $K_{3m} \,\square\, K_{3n}$, the Cartesian product of complete graphs K_{3m} and K_{3n}, as well as good lower and upper bounds of the security number of the Cartesian product of complete graphs with any number of vertices.

Key words: Secure set, Cartesian product.

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1 Introduction and preliminaries

The concept of secure sets was introduced by Brigham, Dutton, and Hedetniemi [3] by restricting defensive alliances [17] to be more "secure". Let us first recall the definition of defensive alliances. Let G be a graph. (Throughout the paper, all graphs considered are simple, finite, and undirected.) A nonempty set $S \subseteq V(G)$ is a *defensive alliance* of G if $|N[x] \cap S| \geq |N[x] - S|$ holds for each $x \in S$, where $N[x]$ denotes the closed neighborhood of x. For secure sets, we ask such a condition also for each subset of S. That is, a non-empty set $S \subseteq V(G)$ is a *secure set* of G if $|N[X] \cap S| \geq |N[X] - S|$ holds for each $X \subseteq S$, where $N[X] = \bigcup_{x \in X} N[x]$. The *security number* of G, denoted $s(G)$, is the size of a minimum secure set of G. The formal definition of a secure set is much longer than the one given in this paper. It requires a clever partitioning of the given attack and defense sets (see [3] for

*The author is also affiliated with the Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia all the details). The definition given above is actually a very nice characterization theorem for secure sets proved by Brigham et al. in the same paper [3]. Its formal statement is given in Theorem 1.

The security graph parameter has been studied extensively after its introduction [1, 2, 4, 5, 6, 19]. (See also [11, 18] for closely related parameters.) A variation of secure sets and the security number called global secure sets and the global security number, respectively, were treated in [12, 13, 14, 15] with an additional condition that the secure set must also dominate the vertices in a graph.

Studies of security in Cartesian product graphs were initiated already in [3], where several upper bounds were determined for grid-like graphs. Afterwards the studies continued in [16] where exact formulae and some other bounds on the security number of grid-like graphs were established. Global secure sets on grid-like graphs were studied in [8, 9, 10]. Strong product graphs were considered in [20], where the security number of grids, cylinders, and toruses was derived. One of the first general results on graph products were given for the lexicographic product of graphs [7]. The statements in the latter article assume at least one factor to be an arbitrary graph. Since there are no results for arbitrary Cartesian product of graphs any step in this direction would be a nice improvement, and this is precisely what we study in this paper. We present two upper bounds of the security number for the Cartesian product of arbitrary graphs. We also investigate a special case where we consider the Cartesian product of complete graphs. In this case, we can determine the security number exactly if each complete graph has order divisible by 3. Such results for special cases were previously known only for the Cartesian product of sparse graphs such as paths and cycles [3, 16].

Note that, in general, it is very hard to determine the security number of a graph. Ho [8] showed that given a set $S \subseteq V(G)$, it is coNP-complete to determine whether S is a secure set of G. The complexity of the problem for determining the security number of a graph was unknown for years and then finally shown to be Σ_2^{P}-complete by Bliem and Woltran [2].

Given two graphs G and H, the *Cartesian product* $G \,\Box\, H$ of G and H has the vertex set $V(G) \times V(H)$ and $(x,y)(x',y') \in E(G \,\Box\, H)$ whenever $x = x'$ and $yy' \in E(H)$, or $xx' \in E(G)$ and $y = y'$. For a vertex $y \in V(H)$ we define the set $G^y = \{(x,y) \in V(G \,\Box\, H) \,|\, x \in V(G)\}$, which is called a *$G$-layer* in the Cartesian product of G and H. For $x \in V(G)$, the *H-layer* ${}^xH$ is defined as ${}^xH = \{(x,y) \in V(G \,\Box\, H) \,|\, y \in V(H)\}$. We may consider G-layers and H-layers as induced subgraphs when appropriate. For a subset $S \subseteq V(G \,\Box\, H)$ we define

$$p_G(S) = \{x \mid (x,y) \in S\} \subseteq V(G)$$

as the *projection set* of S onto the base graph G and

$$p_H(S) = \{y \mid (x,y) \in S\} \subseteq H(G)$$

as the *projection set* of S onto the base graph H.

We finish this section with one of the most important results on secure sets of graphs. The result in Theorem 1 will be heavily used throughout this article.

Theorem 1 (Brigham, Dutton, Hedetniemi [3]) *Set $S \subseteq V(G)$ is a secure set of a graph G if and only if*

$$|N[X] \cap S| \geq |N[X] - S|$$

for all $X \subseteq S$.

Remark 2 *The expression* $|N[X] \cap S| \geq |N[X] - S|$ *is called the* security condition *for* $X \subseteq S$.

2 Upper bounds

We begin this section with an upper bound for the security number of the Cartesian product of two arbitrary graphs.

Proposition 3 *If* G *and* H *are arbitrary connected graphs, then*

$$s(G \,\Box\, H) \leq \min\{s(G)|V(H)|, |V(G)|s(H)\}.$$

Proof. Denote with $V(G) = \{x_1, \ldots, x_m\}$ and $V(H) = \{y_1, \ldots, y_n\}$, $m, n \in \mathbb{N}$, the vertices of graphs G and H, respectively.

Let S be a minimum secure set of G, i.e. $|S| = s(G)$, and let $S' = S \times V(H)$. We prove that S' is a secure set of $G \,\Box\, H$. Suppose that X is an arbitrary subset of S'. For each $i \in \{1, \ldots, n\}$ we define $S_i = S' \cap V(G^{y_i})$ and $X_i = X \cap V(G^{y_i})$. Clearly S_i is a secure set of G^{y_i} for every $i \in \{1, \ldots, n\}$, and hence the inequality

$$|N[X_i] \cap S_i| = |(N[X_i] \cap V(G^{y_i})) \cap S_i| \geq |(N[X_i] \cap V(G^{y_i})) - S_i| = |N[X_i] - S'|$$

is fulfilled for every $i \in \{1, \ldots, n\}$. Before we prove the security condition for the set X, note that

$$|N[X] \cap S'| \geq \sum_{i=1}^{n} |N[X_i] \cap S_i|,$$

since X_i might have some neighbours in other layers than the G^{y_i}-layer. It follows that

$$\begin{aligned} |N[X] \cap S'| &\geq \sum_{i=1}^{n} |N[X_i] \cap S_i| = \sum_{i=1}^{n} |(N[X_i] \cap V(G^{y_i})) \cap S_i| \\ &\geq \sum_{i=1}^{n} |N[X_i] - S'| = |N[X] - S'|. \end{aligned}$$

Since this is true for any subset $X \subseteq S'$, S' must be a secure set of $G \,\Box\, H$, and hence

$$s(G \,\Box\, H) \leq s(G)|V(H)|.$$

Similarly, we can show the following: if S is a minimum secure set of H, then $S' = V(G) \times S$ is a also a secure set of $G \,\Box\, H$, and thus

$$s(G \,\Box\, H) \leq |V(G)|s(H).$$

Clearly, the statement from the proposition follows. $\square$

The upper bound in Proposition 3 is tight in some cases. For example, it is known from [16] that $s(C_m \,\Box\, C_n) = \min\{2m, 2n, 12\}$ for $\max\{m, n\} \geq 4$. Hence, if $1 \leq m, n \leq 6$, then $s(C_m \,\Box\, C_n) = \min\{2m, 2n\} = \min\{s(C_m)|V(C_n)|, |V(C_m)|s(C_n)\}$. However, in general, this bound does not perform so well. Therefore, it is natural to search for a better upper bound. In this sense, we define the notion of an extended secure set.

Definition 4 *Let G be a connected graph. Suppose there exists a minimum secure set $S \subseteq V(G)$ such that $S' = N[S] = S \cup N(S)$ fulfills the following property:*

- *for every subset $X \subseteq (S' - S)$, $|N[X] \cap S| \geq |N[X] - S'|$.*

The set S' is called an extended secure set *of G, and we call the set $S' - S$ the* extended part *of the set S'.*

The condition $|N[X] \cap S| \geq |N[X] - S'|$ in Definition 4 is very similar to the security condition; the difference is that in this condition every subset X from $S' - S$ is protected from the inside, i.e. from the set S. We will later show that not every graph G contains an extended secure set. With this definition we are able to define the extended security number of a graph.

Definition 5 *Let G be a connected graph with at least one extended secure set. We name*

$$es(G) = \min\{|S'| \mid S' \text{ is an extended secure set of } G\}$$

the extended security number *of G.*

The following remark obviously follows directly from the definition of an extended secure set.

Remark 6 *If G is a connected graph with at least one extended secure set, then*

$$s(G) < es(G) \leq |V(G)|.$$

We will call the family of graphs which satisfy the condition in Definition 4 the family of *extended-securable graphs*, and we will denote this family with $\mathcal{G}$. Clearly, not all connected graphs belong to $\mathcal{G}$. Take for example the path on 4 vertices $v_1v_2v_3v_4$, and identify the leaves v_1 an v_4 each with any vertex of a complete graph K_n, $n \geq 3$. The only minimum secure set in this graph is formed by both inner vertices v_2 and v_3. Its closed neighbourhood is the whole path, which means that both leaves v_1 and v_4 belong to the extended part of the secure set. It is easy to check that the vertices v_2 and v_3 cannot protect the leaves v_1 and v_4 against the attack from the vertices of the complete graphs.

One can also show that the set $\mathcal{G}$ is infinite. We can take a similar example as before by taking the path on 6 vertices $v_1v_2v_3v_4v_5v_6$. We may take any two connected graphs H_1 and H_2 with $\delta(H_1), \delta(H_2) \geq 2$, and identify any vertex from H_1 with the leaf v_1, and identify any vertex from H_2 with the leaf v_6. It is again clear that the security number of such a graph is 2. A minimum secure sets is formed by the vertices v_3 and v_4. The closed neighbourhood of this set it $\{v_2, v_3, v_4, v_5\}$, where vertices v_2 and v_5 belong to the extended part of the secure set. Those vertices are attacked only by at most two vertices, v_1 and v_6, and the inner two vertices v_3 and v_4 can easily repel this attack. Hence, this graph belongs to the family $\mathcal{G}$, and since we can chose H_1 and H_2 almost arbitrarily, there are infinitely many graphs that lie in $\mathcal{G}$. The construction of such graphs was based on the assumption that the security number is 2. One can find many more examples, for larger values of the security number, that lie in $\mathcal{G}$. We are now ready to prove the following theorem.

Theorem 7 *Let G and H be two arbitrary graphs from $\mathcal{G}$ and let S_1 and S_2 be minimum secure sets of G and H, respectively, that satisfy the following two conditions:*

1. *$S_1' = N[S_1]$ is a minimum extended secure set of G,*
2. *$S_2' = N[S_2]$ is a minimum extended secure set of H.*

Moreover, let $A = S_1 \times S_2$, $B = S_1 \times (S_2' - S_2)$ and $C = (S_1' - S_1) \times S_2$. If

$$|N[X \cap B] \cap N[X \cap C] \cap A| \leq |(N[X \cap B] \cap N[X \cap C]) - A|,$$

holds true for any subset $X \subseteq A \cup B \cup C$, then

$$s(G \,\Box\, H) \leq s(G)s(H) + s(G)(es(H) - s(H)) + (es(G) - s(G))s(H),$$

and the bound is sharp.

Proof. Let $V(G) = \{x_1, \ldots, x_m\}$ and $V(H) = \{y_1, \ldots, y_n\}$, $m, n \in \mathbb{N}$, be the vertices of graphs G and H, respectively. Without loss of generality let us assume $S_1 = \{x_1, \ldots, x_{k_1}\}$, $S_1' - S_1 = \{x_{k_1+1}, \ldots, x_{k_2}\}$, $1 \leq k_1 < k_2 \leq m$, and $S_2 = \{y_1, \ldots, x_{\ell_1}\}$, $S_2' - S_2 = \{y_{\ell_1+1}, \ldots, x_{\ell_2}\}$, $1 \leq \ell_1 < \ell_2 \leq n$.

Additionally to the sets A, B and C we define the sets

$$\begin{aligned}
D &= (N[B] \cap (S_1 \times V(H))) - (A \cup B), \\
E &= (N[B] \cap N[C]) - A, \\
F &= (N[C] \cap (V(G) \times S_2)) - (A \cup C).
\end{aligned}$$

For the visualisation of all these sets see Figure 1. We will show that $S = A \cup B \cup C$ is a secure set of $G \,\Box\, H$. The sets A, B and C form the defenders and the sets D, E and F are the attackers of the set S. Take an arbitrary subset $X \subseteq S$. We define the sets $X_A = X \cap A$, $X_B = X \cap B$ and $X_C = X \cap C$.

According to Figure 1, X_A is not attacked at all, therefore we need to show that X_B and X_C can be both protected at the same time. We first turn our attention to X_B. For every $j \in \{\ell_1 + 1, \ldots, \ell_2\}$ we have

$$|(N[X_B] \cap V(G^{y_j})) \cap B| \geq |(N[X_B] \cap V(G^{y_j})) - B|,$$

since $V(G^{y_j}) \cap B$ is by assumption a secure set of the G^{y_j}-layer. Moreover, for every $i \in \{1, \ldots, k_1\}$ we also have

$$|(N[X_B] \cap V({}^{x_i}H)) \cap A| \geq |(N[X_B] \cap V({}^{x_i}H)) - (A \cup B)|,$$

since $V({}^{x_i}H) \cap (A \cup B)$ is by assumption an extended secure set of the ${}^{x_i}H$-layer. We are ready to prove that the set X_B can be protected:

$$\begin{aligned}
|N[X_B] \cap S| &\geq \sum_{j=\ell_1+1}^{\ell_2} |(N[X_B] \cap V(G^{y_j})) \cap B| + \sum_{i=1}^{k_1} |(N[X_B] \cap V({}^{x_i}H)) \cap A| \\
&\geq \sum_{j=\ell_1+1}^{\ell_2} |(N[X_B] \cap V(G^{y_j})) - B| + \sum_{i=1}^{k_1} |(N[X_B] \cap V({}^{x_i}H)) - (A \cup B)| \\
&= |N[X_B] - S|.
\end{aligned}$$

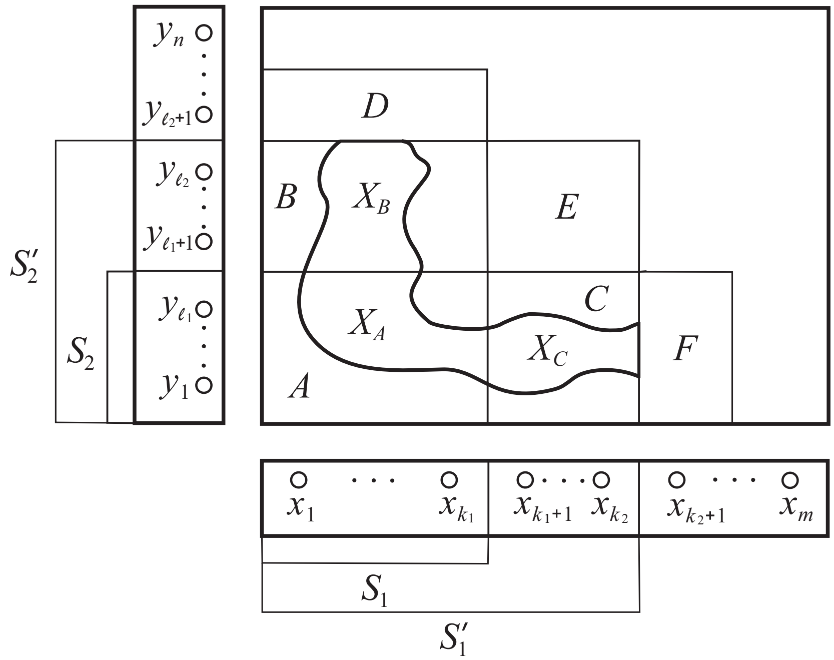

Figure 1: The sets A, B, C, D, E, F and the sets X_A, X_B, X_C in the graph $G \,\square\, H$.

The calculation shows that the set X_B can be horizontally protected against the vertices from E with the help of the vertices from B because S_1 is a secure set, and it can also be vertically protected against the vertices from D with the help of the vertices from A because S_2' is an extended secure set. By symmetry, we can show with a similar proof than above that X_C can be vertically protected against the vertices from E with the help of the vertices from C because S_2 is a secure set, and it can also be horizontally protected against the vertices from F with the help of the vertices from A because S_1' is an extended secure set. The problem that remains is that vertices from A are counted twice for protecting both sets X_B and X_C, and the same is true for the vertices in E, which are also counted twice for attacking both sets X_B and X_C. Hence we need to show that both sets X_B and X_C can still be protected at the same time.

We denote with a, b and c the number of the vertices from A, B and C, respectively, that protect the vertices from X. Additionally, we assume that a_1 is the number of the vertices that can protect only the vertices from X_B, a_3 the number of the vertices that can protect only the vertices from X_C, and a_2 the number of the vertices that can protect vertices from both X_B and X_C. It is clear that $a = a_1 + a_2 + a_3$, $a_1, a_2, a_3 \geq 0$. As for the attackers, we denote with d, e and f the number of the vertices from D, E and F, respectively, that attack the vertices from X. Additionally, we assume that e_1 is the number of the vertices that can attack only the vertices from X_B, e_3 the number of the vertices that can attack only the vertices from X_C, and e_2 the number of the vertices that can attack vertices from both X_B and X_C. Again, it is clear that $e = e_1 + e_2 + e_3$, $e_1, e_2, e_3 \geq 0$. We first note that $|N[X \cap B] \cap N[X \cap C] \cap A| = a_2$ and $|(N[X \cap B] \cap N[X \cap C]) - A| = e_2$. According to the assumption $|N[X \cap B] \cap N[X \cap C] \cap A| \leq |(N[X \cap B] \cap N[X \cap C]) - A|$ from the theorem we have $a_2 \leq e_2$. To end the proof we need to show that $a + b + c \geq d + e + f$ or equivalently

$(a + b + c) - (d + e + f) \geq 0$.

Since we already know that X_B and X_C can each separately be protected, we have $a_1 + a_2 + b \geq d + e_1 + e_2$ and $a_2 + a_3 + c \geq e_2 + e_3 + f$. We sum up both inequalities:

$$\begin{aligned} a_1 + 2a_2 + a_3 + b + c &\geq d + e_1 + 2e_2 + e_3 + f \\ a + b + c + a_2 &\geq d + e + f + e_2 \\ (a + b + c) - (d + e + f) &\geq e_2 - a_2 \geq 0 \end{aligned}$$

where in the last inequality we used the fact that $a_2 \leq e_2$. It follows that

$$s(G \,\Box\, H) \leq |A \cup B \cup C| = s(G)s(H) + s(G)(es(H) - s(H)) + (es(G) - s(G))s(H). \quad (1)$$

To show the sharpness of the bound we take $G = C_m$, and $H = C_n$, where $V(C_m) = \{x_1, \ldots, x_m\}$, $x_i x_{i+1} \in E(C_m)$ for all i (modulo m), $V(C_n) = \{y_1, \ldots, y_n\}$, $y_j y_{j+1} \in E(C_n)$ for all j (modulo n), and $m, n \geq 6$. The sets $S_1 = \{x_2, x_3\}$ and $S_2 = \{y_2, y_3\}$ are secure sets of graphs C_m and C_n, respectively. Both graphs also belong to the family $\mathcal{G}$, where $S_1' = \{x_1, x_2, x_3, x_4\}$ and $S_2' = \{y_1, y_2, y_3, y_4\}$ are the corresponding extended secure sets. Therefore, $s(C_m) = s(C_n) = 2$ and $es(C_m) = es(C_n) = 4$. It is also very easy to check the condition $|N[X \cap B] \cap N[X \cap C] \cap A| \leq |(N[X \cap B] \cap N[X \cap C]) - A|$, since both sides of the inequality are always equal for any subset $X \subseteq A \cup B \cup C$. By (1), we have

$$s(C_m \,\Box\, C_n) \leq s(C_m)s(C_n) + s(C_m)(es(C_n) - s(C_n)) + (es(C_m) - s(C_m))s(C_n) = 4 + 4 + 4 = 12.$$

We already know from [16] that $s(C_m \,\Box\, C_n) = 12$, $m, n \geq 6$. $\square$

3 The Cartesian product of complete graphs

The bound in Theorem 7 can be also very bad. If we take for example the complete graphs K_m and K_n, which also belong to the family $\mathcal{G}$, we have $es(K_m) = m$ and $es(K_n) = n$, and the bound in Theorem 7 becomes even worse than the bound in Proposition 3. Therefore, we turn our attention to the Cartesian product of complete graphs and derive a better upper bound for them. For some values of m and n we even prove exact results. The following theorem gives a better upper bound for the Cartesian product of complete graphs.

Theorem 8 *Let $m, n \geq 1$. Then*

$$s(K_m \,\Box\, K_n) \leq \min\{ab \,|\, ab \geq ad + bc\},$$

where $a \in \{1, \ldots, m\}$, $b \in \{1, \ldots, n\}$, $c = m - a$ and $d = n - b$.

Proof. Let $a \in \{1, \ldots, m\}$ and $b \in \{1, \ldots, n\}$ be such values that $ab \geq ad + bc$ holds true, where $c = m - a$ and $d = n - b$. Let $V(K_m) = \{x_1, \ldots, x_a, \ldots, x_m\}$ and $V(K_n) = \{y_1, \ldots, y_b, \ldots, y_n\}$ be the vertices of graphs K_m and K_n, respectively (note that a might be 1 or m, and b might be 1 or n). We will show that

$$S = \{x_1, \ldots, x_a\} \times \{y_1, \ldots, y_b\}$$

is a secure set of $K_m \Box K_n$. Let $X \subseteq S$ be an arbitrary subset of S, and let $X_1 = p_{K_m}(X)$ and $X_2 = p_{K_n}(X)$ be the projection sets of X onto both factor graphs K_m and K_n, respectively. Clearly it can happen that the sets X_1 and/or X_2 are nonconsecutive. Since the Cartesian product $K_m \Box K_n$ is formed from complete graphs, we can always rearrange the vertices of $\{x_1, \ldots, x_a\}$ and/or $\{y_1, \ldots, y_b\}$ in such a way that the number of attackers and defenders of every subset $X \subseteq S$ will remain the same, but the corresponding sets X_1 and X_2 will be consecutive. After this rearrangement, we define $a_1 = |X_1|$, $b_1 = |X_2|$, and $a_2 = a - a_1$, $b_2 = b - b_1$ (see Figure 2).

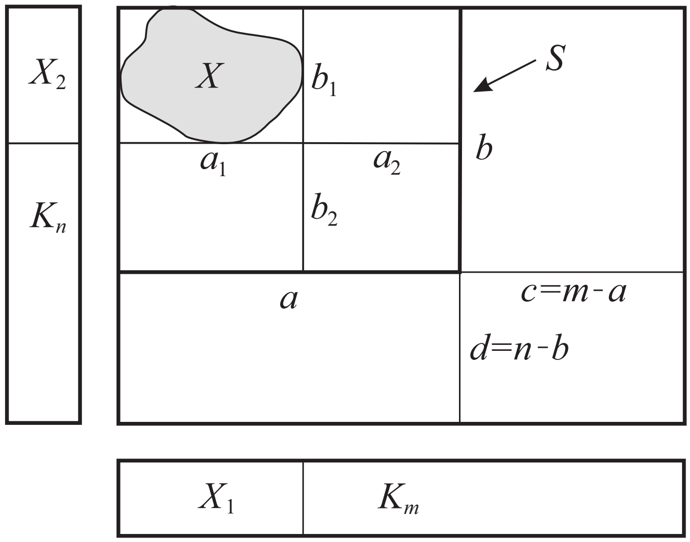

Figure 2: The sets S, X, X_1 and X_2 in the graph $K_m \Box K_n$.

We have $|N[X] \cap S| = a_1b_1 + a_1b_2 + a_2b_1$ and $|N[X] - S| = a_1d + b_1c$. We have to show that $a_1b_1 + a_1b_2 + a_2b_1 = |N[X] \cap S| \geq |N[X] - S| = a_1d + b_1c$.

Suppose the opposite that $a_1b_1 + a_1b_2 + a_2b_1 < a_1d + b_1c$. Since $ab = a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2$ and $ad + bc = a_1d + a_2d + b_1c + b_2c$, we can rewrite the inequality $ab \geq ad + bc$ as follows:

$$a_1b_1 + a_1b_2 + a_2b_1 + a_2b_2 \geq a_1d + a_2d + b_1c + b_2c.$$

Using $a_1b_1 + a_1b_2 + a_2b_1 < a_1d + b_1c$, we get

$$\begin{aligned} a_1d + b_1c + a_2b_2 &> a_1d + a_2d + b_1c + b_2c \\ a_2b_2 &> a_2d + b_2c. \end{aligned}$$

Since a_2, b_2, c and d are positive integers, it follows from the last inequality that $a_2b_2 > a_2d$ and $a_2b_2 > b_2c$. The first inequality yields $b_2 > d$, and the second one yields $a_2 > c$. Thus,

$$a_1b_1 + a_1b_2 + a_2b_1 > a_1b_2 + a_2b_1 > a_1d + b_1c,$$

which is a contradiction, since we assumed that $a_1b_1 + a_1b_2 + a_2b_1 < a_1d + b_1c$. We proved that ab is an upper bound of $s(K_m \Box K_n)$ for any $a \in \{1, \ldots, m\}$, $b \in \{1, \ldots, n\}$, whenever $ab \geq ad + bc$ ($c = m - a$ and $d = n - b$) is fulfilled. It follows that $s(K_m \Box K_n) \leq \min\{ab \mid ab \geq ad + bc\}$. $\Box$

We will call the set $S = \{x_1, \dots, x_a\} \times \{y_1, \dots, y_b\}$ defined in the proof of Theorem 8 a *rectangular set* of the graph $K_m \,\Box\, K_n$. Theorem 8 also shows that for any rectangular set S of $K_m \,\Box\, K_n$ one does not need to check the security condition $|N[X] \cap S| \geq |N[X] - S|$ for all subsets $X \subseteq S$ in order for S to be a secure set. Namely, it is enough to check the security condition only for the set S. We summarize this thoughts into the following corollary.

Corollary 9 *Let $m, n \geq 1$, and let $V(K_m) = \{x_1, \dots, x_a, \dots, x_m\}$, $1 \leq a \leq m$, and $V(K_n) = \{y_1, \dots, y_b, \dots, y_n\}$, $1 \leq b \leq n$, be the vertices of K_m and K_n, respectively. The rectangular set $S = \{x_1, \dots, x_a\} \times \{y_1, \dots, y_b\}$ is a secure set of $K_m \,\Box\, K_n$ if and only if $|S| \geq |N[S] - S|$. Moreover, if $S = \{x_1, \dots, x_a\} \times \{y_1, \dots, y_b\}$ is a secure set of $K_m \,\Box\, K_n$, it follows that $a \geq \frac{m+1}{3}$ and $b \geq \frac{n+1}{3}$.*

Proof. Let $S = \{x_1, \dots, x_a\} \times \{y_1, \dots, y_b\}$ rectangular set, where $c = m - a$ and $d = n - b$. If S is a secure set of $K_m \,\Box\, K_n$, then it is obvious that $|S| = |N[S] \cap S| \geq |N[S] - S|$.

For the converse, suppose that $|S| \geq |N[X] - S|$ holds, which is equivalent to $ab \geq ad + bc$. By Theorem 8, S must be a secure set.

For the second part of the corollary, we again use $ab \geq ad + bc$:

$$\begin{aligned} ab &\geq ad + bc = a(n-b) + b(m-a) = an - ab + bm - ab \\ 3ab &\geq an + bm \\ a &\geq \frac{an}{3b} + \frac{m}{3} \geq \frac{n}{3n} + \frac{m}{3} = \frac{m+1}{3}. \end{aligned}$$

Similarly, we can derive

$$\begin{aligned} 3ab &\geq an + bm \\ b &\geq \frac{n}{3} + \frac{bm}{3a} \geq \frac{n}{3} + \frac{m}{3m} = \frac{n+1}{3}. \end{aligned}$$

$\square$

For some special values of m and n in $K_m \,\Box\, K_n$ we can prove that the upper bound in Theorem 8 is tight.

Theorem 10 *Let $m, n \geq 1$. Then $s(K_{3m} \,\Box\, K_{3n}) = 4mn$.*

Proof. Using Theorem 8 and choosing $a = 2m$, $b = 2n$, $c = 3m - 2m = m$, $d = 3n - 2n = n$, we get

$$4mn = ab \geq ad + bc = 4mn.$$

Hence, by Theorem 8 we have $s(K_{3m} \,\Box\, K_{3n}) \leq 4mn$.

To prove the equality, we choose an arbitrary minimum secure set S of $K_{3m} \,\Box\, K_{3n}$ with $|S| \leq 4mn$. Let $a = |p_{K_{3m}}(S)|$ be the size of the projection set of S onto the base graph K_{3m}. Similarly, let $b = |p_{K_{3n}}(S)|$ be the size of the projection set of S onto the base graph K_{3n}. Thus $|S| = ab - k$ for some $k \geq 0$. It is easy to see that $|N[S] \cap S| = |S| = ab - k$ and $|N[S] - S| = (3m - a)b + (3n - b)a + k$ (see Figure 3). Since S is a secure set of $K_{3m} \,\Box\, K_{3n}$, it follows that $|S| = |N[S] \cap S| \geq |N[S] - S|$, and hence $ab \geq mb + na + \frac{2k}{3}$.

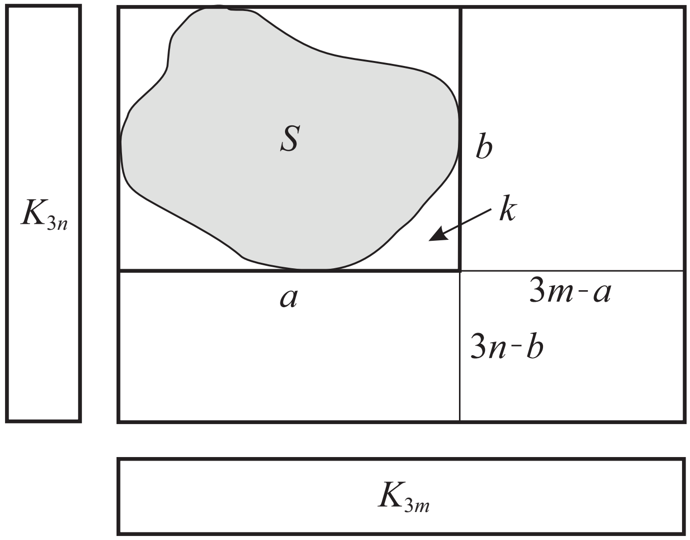

Figure 3: The set S in the graph $K_{3m} \square K_{3n}$.

Firstly suppose that $a \leq m$. Then

$$mb + na + \frac{2k}{3} > mb \geq ab,$$

which is a contradiction. Similarly, if $b \leq n$, then

$$mb + na + \frac{2k}{3} > na \geq ba,$$

which is again a contradiction. Henceforth, we can assume that $a > m$ and $b > n$.

Now we prove that $|S| = 4mn$. The following inequalities are equivalent:

$$\begin{aligned} ab &\geq mb + na + \frac{2k}{3} \\ ab - na &\geq mb + \frac{2k}{3} \\ a(b-n) &\geq mb + \frac{2k}{3} \\ a &\geq \frac{mb}{b-n} + \frac{2k}{3(b-n)}. \end{aligned}$$

Note that the last inequality does not turn because $b > n$. We continue with the next set of equivalent inequalities:

$$
\begin{aligned}
a &\geq \frac{mb}{b-n} + \frac{2k}{3(b-n)} \\
ab &\geq \frac{mb^2}{b-n} + \frac{2kb}{3(b-n)} \\
ab - k &\geq \frac{mb^2}{b-n} + \frac{2kb}{3(b-n)} - \frac{3k(b-n)}{3(b-n)} \\
ab - k &\geq \frac{mb^2}{b-n} + \frac{2kb - 3kb + 3kn}{3(b-n)} \\
ab - k &\geq \frac{mb^2}{b-n} + \frac{k(3n-b)}{3(b-n)} \\
4mn \geq |S| = ab - k &\geq \frac{mb^2}{b-n} + \frac{k(3n-b)}{3(b-n)} \geq \frac{mb^2}{b-n} + 0.
\end{aligned}
$$

The last inequality is true because $n < b \leq 3n$. We get

$$
4mn \geq \frac{mb^2}{b-n},
$$

which is equivalent to

$$
(b-2n)^2 \leq 0.
$$

The only logical solution to this inequality is $b = 2n$. We continue with a similar calculation for a:

$$
\begin{aligned}
ab &\geq mb + na + \frac{2k}{3} \\
ab - mb &\geq na + \frac{2k}{3} \\
b(a-m) &\geq na + \frac{2k}{3} \\
b &\geq \frac{na}{a-m} + \frac{2k}{3(a-m)} \\
ab &\geq \frac{na^2}{a-m} + \frac{2ka}{3(a-m)} \\
ab - k &\geq \frac{na^2}{a-m} + \frac{2ka}{3(a-m)} - \frac{3k(a-m)}{3(a-m)} \\
ab - k &\geq \frac{na^2}{a-m} + \frac{2ka - 3ka + 3km}{3(a-m)} \\
ab - k &\geq \frac{na^2}{a-m} + \frac{k(3m-a)}{3(a-m)} \\
4mn \geq |S| = ab - k &\geq \frac{na^2}{a-m} + \frac{k(3m-a)}{3(a-m)} \geq \frac{na^2}{a-m} + 0 \\
4mn &\geq \frac{na^2}{a-m} \\
(a-2m)^2 &\leq 0.
\end{aligned}
$$

We get that $a = 2m$ is the only logical solution to this inequality. It follows that $|S| = 4mn - k$. Since the inequality $ab \geq mb + na + \frac{2k}{3}$ still need to be fulfilled, we get $4mn \geq 4mn + \frac{2k}{3}$, which is only possible for $k = 0$. Thus $|S| = 4mn$ and $s(K_{3m} \,\square\, K_{3n}) = 4mn$. $\square$

For all other values of m and n, which are not congruent 0 modulo 3, we can give a lower and an upper bound for the security number of the graph $K_m \,\square\, K_n$.

Proposition 11 *Let $m, n \geq 1$ and $k, \ell \in \{0, 1, 2\}$. Then*

$$4mn \leq s(K_{3m+k} \,\square\, K_{3n+\ell}) \leq 4(m+1)(n+1).$$

Proof. The upper bound follows from Theorem 8 by setting $a = 2(m+1)$, $b = 2(n+1)$, $c = (3m+k) - a$, and $d = (3n+\ell) - b$, which gives

$$\begin{aligned} ab - ad - bc &= 2(m+1) \cdot 2(n+1) - 2(m+1)(n+\ell-2) - 2(n+1)(m+k-2) \\ &= 2((m+1)(3-\ell) + (3-k)(n+1)) > 0. \end{aligned}$$

Let S be a minimum secure set of $K_{3m+k} \,\square\, K_{3n+\ell}$. Observe that $|N[S]| \geq \max\{|p_{K_{3m+k}}(S)| \cdot (3n+\ell), (3m+k) \cdot |p_{K_{3n+\ell}}(S)|\}$. Hence, if $|p_{K_{3m+k}}(S)| \geq 3m$ or $|p_{K_{3n+\ell}}(S)| \geq 3n$, then $|N[S]| \geq 9mn$, and thus $|S| \geq \frac{9mn}{2} > 4mn$. Assume that $|p_{K_{3m+k}}(S)| < 3m$ and $|p_{K_{3n+\ell}}(S)| < 3n$. By removing k vertical and ℓ horizontal layers that do not intersect S, we can get an induced subgraph H of $K_{3m+k} \,\square\, K_{3n+\ell}$ such that H is isomorphic to $K_{3m} \,\square\, K_{3n}$ and $S \subseteq V(H)$. We can see that S is a secure set of H since any subset $X \subseteq S$ has more attackers in $K_{3m+k} \,\square\, K_{3n+\ell}$ than it has in H. Therefore,

$$s(K_{3m+k} \,\square\, K_{3n+\ell}) = |S| \geq s(H) = s(K_{3m} \,\square\, K_{3n}) = 4mn.$$

$\square$

Corollary 12 *Let $n \geq 1$. Then*

$$\lim_{n \to \infty} \frac{s(K_n \,\square\, K_n)}{|V(K_n \,\square\, K_n)|} = \frac{4}{9}.$$

Proof. By Proposition 11 we have $4n^2 \leq s(K_{3n+k} \,\square\, K_{3n+\ell}) \leq 4(n+1)^2$ for all $k, \ell \in \{0, 1, 2\}$. Since $|V(K_{3n+k})| = 3n + k$ and $|V(K_{3n+\ell})| = 3n + \ell$, we have

$$\frac{4n^2}{9n^2 + 3n\ell + 3nk + k\ell} \leq \frac{s(K_{3n+k} \,\square\, K_{3n+\ell})}{|V(K_{3n+k})||V(K_{3n+\ell})|} \leq \frac{4(n+1)^2}{9n^2 + 3n\ell + 3nk + k\ell}.$$

When n tends to infinity, we get

$$\frac{4}{9} \leq \lim_{n \to \infty} \frac{s(K_{3n+k} \,\square\, K_{3n+\ell})}{|V(K_{3n+k} \,\square\, K_{3n+\ell})|} \leq \frac{4}{9}$$

for all $k, \ell \in \{0, 1, 2\}$. $\square$

4 Concluding remarks

It would be interesting to find at least one non-trivial lower bound for the security number of the Cartesian product of arbitrary graphs. It is however clear that this would be considerably more difficult than proving the upper bounds which are usually given by constructions. Nevertheless, our investigations lead us to propose a Vizing-type inequality for the security number as an open problem.

Problem 13 *Is $s(G \,\Box\, H) \geq s(G)s(H)$ true for arbitrary connected graphs G and H?*

A simpler inequality to the one above could be $s(G \,\Box\, H) \geq \max\{s(G), s(H)\}$, but this one turned out to be challenging to prove as well.

In this paper a construction for an infinite family of extended-securable graphs with the security number 2 is given. Since the characterization of graphs with the security number 2 is very well known (see [3]), it almost seems that the extended-securable graphs with the security number 2 constructed in this paper are the only such graphs. Hence, it would be appropriate to pose the following problem.

Problem 14 *Let k be a positive integer. Characterize graphs from the family of extended-securable graphs $\mathcal{G}$ with the security number equal to k.*

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