# Indicated coloring game on Cartesian products of graphs

Boštjan Brešar[a,b] Marko Jakovac[a,b] Daša Štesl[a]

[a] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

**Abstract**

Indicated coloring game is played on a simple graph $G$ by two players, and a fixed set $C$ of colors. In each round of the game Ann indicates an uncolored vertex, and Ben colors it using a color from $C$, obeying just the proper coloring rule. The goal of Ann is to achieve a proper coloring of the whole graph, while Ben is trying to prevent this. The minimum cardinality of the set of colors $C$ for which Ann has a winning strategy is called the indicated chromatic number, $\chi_i(G)$, of a graph $G$. In this paper, we prove that the indicated chromatic number of the Cartesian product $G \,\square\, K_{n,m}$ is equal to 3 if and only if $\chi_i(G) = 3$. We also prove that $\chi_i(G \,\square\, T) = \chi(G)$, where $G$ is a block graph and $T$ is a tree. Indicated colorings in some other classes of Cartesian products of graphs are also studied. The investigations lead us to propose a Sabidussi-type equality as an open problem.

**Key words**: indicated chromatic number, game coloring, Cartesian product

**AMS subject classification (2010)**: 05C57, 05C15, 05C76

## 1 Introduction

The study of coloring games in graphs was initiated independently by Gardner [11] and Bodlander [5]. The initial version of the coloring game triggered numerous investigations, which resulted in the development of various methods and strategies [4]. Interesting connections between the game coloring numbers and some established graph invariants were found [7, 8, 14]. In addition, several variations of the coloring game were introduced [1, 3, 6, 15, 17]; see a recent survey on coloring games [21], a dynamic survey on combinatorial games [9], and the corresponding references therein.

In this paper, we consider a variation of the coloring game, which was introduced by Grzesik under the name *indicated coloring game* [12] (the idea for this game is contributed to Grytczuk). It is played on a simple graph $G$ by two players, Ann and Ben, with a fixed set of available colors. Each round in the game consists of Ann selecting a previously unselected vertex and Ben assigning to that vertex a color, which has not been assigned to any of its neighbors (that is, the coloring must be proper). Players have opposite goals, Ann wants to achieve a (proper) coloring of all vertices, while Ben wants to prevent this to happen (i.e., his goal is to eventually have an uncolored vertex such that all available colors appear in its neighborhood). The minimum number of available colors that Ann needs to win the game on a graph $G$, regardless of how Ben plays, is called the *indicated chromatic number* of $G$, denoted $\chi_i(G)$.

Clearly, $\chi_i(G)$ is bounded from above by $\Delta(G)+1$, and it is easy to see that in bipartite graphs $G$, $\chi_i(G)=2$. However, one of the basic questions that are usually posed for invariants of this type, whether for any graph $G$ the indicated chromatic number $\chi_i(G)$ is bounded by a function $f$ dependent only on the chromatic number $\chi(G)$, is still open. It is shown in [12] that there are graphs $G$ with $\chi_i(G) \geq \frac{4}{3}\chi(G)$, yet it is suspected that such a function $f$ exists. Grzesik also proposed an innocently looking question whether more available colors may only help Ann to win, that is, whether Ann can win the game with $k$ available colors as soon as $k \geq \chi_i(G)$. The question was investigated in [18, 19], where it was answered in the affirmative for various classes of graphs. In addition, in [10] the positive answer to the question was given for several families of Cartesian products of graphs. (We remark that a version of the indicated chromatic number was considered in matroids and was shown to be equal to the chromatic number [16].)

Given two graphs $G$ and $H$, the *Cartesian product* $G \,\Box\, H$ of $G$ and $H$ has the vertex set $V(G)\times V(H)$ and $(g,h)(g',h') \in E(G \,\Box\, H)$ whenever $g=g'$ and $hh' \in E(H)$, or, $gg' \in E(G)$ and $h=h$. The Cartesian product is arguably the most investigated graph product with many applications and relations to different topics [13]. For the chromatic number, a classical result by Sabidussi [20] states that $\chi(G \,\Box\, H) = \max\{\chi(G), \chi(H)\}$. As shown in [2, 22], the game chromatic number behaves much less tamely under the Cartesian product of graphs. On the other hand, for the indicated chromatic number of Cartesian products, as studied in [10], a large majority of results assert that $\chi_i(G_1 \,\Box\, G_2) = \chi(G_1 \,\Box\, G_2)$, where for all considered factor graphs $G_k$, $\chi_i(G_k) = \chi(G_k)$. Additional example was presented in [10] showing that $\chi_i(D \,\Box\, K_2) > \chi(D \,\Box\, K_2)$, however, the corresponding graph $D$ has $\chi_i(D) > \chi(D)$.

In this paper, we further investigate the indicated chromatic number of Cartesian products of graphs. We prove that for a graph $G$, $\chi_i(G)=3$ if and only if $\chi_i(G \,\Box\, K_{m,n}) = 3$ (where $K_{m,n}$ is the complete bipartite graph). The result is in a sense stronger than previous results on this topic, because it does not rely on the structure of a graph $G$ (since it is not known which graphs $G$ have $\chi_i(G)=3$), but only on the property that their indicated chromatic number is 3. In addition, we prove that if $G$ is a block graph and $T$ a tree, then $\chi_i(G \,\Box\, T) = \chi(G)$, which generalizes the result from [10] about the Cartesian products of cliques and trees. Finally, we consider the indicated chromatic number of the Cartesian product of two cycles (at least one of which is not even), and show that $\chi_i(C_3 \,\Box\, C_n) = 3$ for all integers $n \geq 3$. While we believe that the indicated chromatic number of a non-bipartite Cartesian product of two cycles is always 3, we could not prove it in all detail due to enormous technical difficulties. Instead, we present a strategy of Ann, which we believe serves the purpose of proving it, and leave it as an open problem.

All results in this paper and in [10] give the equality

$$\chi_i(G_1 \,\Box\, G_2) = \max\{\chi_i(G_1), \chi_i(G_2)\},$$

and we suspect that this could be true for all pairs of graphs $G_1$ and $G_2$, which would give a Sabidussi-type result, where the indicated chromatic number replaces the chromatic number.

In the next section, we first give some necessary definitions and establish the notation. We continue with all the mentioned results and their proofs. In the last section, we give some concluding remarks. In particular, we present Ann's strategy for which we believe it enables her to win in $C_n \,\square\, C_m$ for all $m$ and $n$ when the game is played with 3 available colors. We also pose some open problems.

## 2 Results

We begin this section with some more definitions.

Given two graphs $G$ and $H$, and a vertex $y \in V(H)$, the set $G^y = \{(x, y) \in V(G \,\square\, H) : x \in V(G)\}$ is called *a $G$-fiber* in the Cartesian product of $G$ and $H$. For $x \in V(G)$, the *$H$-fiber* ${}^xH$ is defined as ${}^xH = \{(x, y) \in V(G \,\square\, H) : y \in V(H)\}$. We may consider $G$-fibers and $H$-fibers as induced subgraphs when appropriate. The *projection* to $G$ is the map $p_G : V(G \,\square\, H) \to V(G)$ defined by $p_G(x, y) = x$. We let $[k] = \{1, \dots, k\}$.

Recall that a *block* of a graph $G$ is a maximal connected subgraph of $G$, which has no cut vertices (that is, a maximal 2-connected subgraph or a $K_2$ whose edge is a cut-edge of $G$). A graph in which each block is a complete graph, is called a *block graph*.

**Definition 1** *Suppose that Ann and Ben play an indicated coloring game on a graph $G$ using a fixed set of colors $C$. If at some point in the game Ann selects a vertex $x \in V(G)$ for which Ben can use only one color from $C$, then we say that $x$ is a* fixed vertex *or that a* color is fixed *for $x$.*

**Lemma 2** *Let $G, H$ be connected graphs. If $\chi_i(G \,\square\, H) \le k$, then $\chi_i(G) \le k$ for each positive integer $k$.*

**Proof.** We show the contrapositive statement, that $\chi_i(G) > k$ implies $\chi_i(G \,\square\, H) > k$.

First, if $\chi(H) > k$, then $\chi_i(G \,\square\, H) \ge \chi(G \,\square\, H) = \max\{\chi(G), \chi(H)\} > k$. Next supppose $\chi(H) = l \le k$. This means there is a proper coloring of $H$, say $c : V(H) \to [k]$, using $k$ colors. Let $B_1, B_2, \dots, B_k$ be the color classes of this coloring, such that $c(v) = i$ for $v \in B_i$, where $i \in [k]$. By the assumption, Ben has a winning strategy in $G$ when $k$ colors are available. Now, the indicated coloring game is played on $G \,\square\, H$ with $k$ colors.

Ben will imagine that simultaneously an indicated coloring game is played on $G$ by using $k$ colors, denoted $\alpha_1 = (1, 2, \dots, k), \alpha_2 = (2, 3, \dots k, 1), \dots, \alpha_{k-1} = (k-1, k, 1, 2, \dots, k-2), \alpha_k = (k, 1, 2, \dots, k-1)$. Whenever Ann will select a vertex $(v, y_i)$ such that no vertex from ${}^vH$ has been selected before, Ben will consider this as a selection of vertex $v \in V(G)$ by Ann in the imagined game. He will color $v$ by one of the colors $\alpha_1, \dots, \alpha_k$ according to his strategy in $G$. At the same time, he will copy the corresponding coordinate of that color to the original game in the following way: if Ann selected $(v, y)$ for some $y \in B_i$ (and no vertex from ${}^vH$ has been selected before), and $\alpha_n$, where $n \in [k]$, is the color in the imagined game on $G$ that Ben uses according to his strategy, then he will color $(v, y)$ with the $i$-th coordinate of $\alpha_n$.

On the other hand, if Ann selects a vertex $(v, z)$, and $(v, y)$ has been selected earlier for some $y \ne z$, then in the imagined game on $G$ Ben does nothing, because he already colored $v$ by one of the colors from $\{\alpha_1, \dots, \alpha_k\}$ in this game. He only colors $(v, z)$ with the corresponding coordinate ($j$th coordinate if $z \in B_j$) of the color $\alpha_n$ he used in the imagined game for $v \in V(G)$.

With this strategy, the coloring of vertices in a fiber ${}^vH$ will be proper, and will be (pre)determined by the imagined game in $G$, as soon as one of these vertices is picked by Ann. However, since Ben wins the game on $G$ with $k$ colors, this means that there will be a vertex $v$ in the imagined game on $G$, which will have in the neighborhood all $k$ colors $\alpha_1, \ldots,$ $\alpha_k$. This implies that in some $G^y$ (in whichever Ann will select the corresponding neighbors of $(v, y)$ first) there will be a vertex in ${}^vH$ such that $v$ will have all $k$ colors $1, 2 \ldots, k$ in its neighborhood before it will be selected by Ann. Hence, $\chi_i(G \,\square\, H) > k$. $\square$

**Theorem 3** *If $G$ is a graph, then $\chi_i(G) = 3$ if and only if $\chi_i(G \,\square\, K_2) = 3$.*

**Proof.** Let $G$ be a graph, $V(G) = \{v_1, \ldots, v_n\}$ and $V(K_2) = \{x, y\}$.

First suppose that $\chi_i(G) = 3$. Hence, Ann has a winning strategy in $G$ when three colors are available. Ann can use the same winning strategy in the fiber $G^x$ of $G$, which is isomorphic to $G$, by which $G^x$ is properly colored by colors from $\{1, 2, 3\}$. In the remainder, we present the strategy of Ann by which she selects vertices in $G^y$. This will be done in several steps, in which a partition of $G^y$ into sets $U_i$ will be built. Let $U_1 = \emptyset$. Now, Ann picks any vertex from $G^y$, say $(v_1, y)$, and Ben colors it with any color available. (Note that Ben has two colors to choose from.) At the same time, $U_1 = \{(v_1, y)\}$. In the next moves Ann picks all neighbors of $(v_1, y)$ with a fixed color, and adds those vertices to $U_1$. Ann continues to pick an arbitrary vertex of $U_1$ that still has some neighbors in $G^y \backslash U_1$ that are fixed, and adds them to $U_1$. She continues with this procedure until there are no more vertices in $U_1$ with a fixed neighbor in $G^y \backslash U_1$. If $U_1 = V(G)$, then all vertices of $G$ have been properly colored, and the proof is done. Otherwise, $U_1 \subsetneq G^y$. Let $U_2 = \emptyset$. Ann picks an arbitrary vertex $(v_2, y)$ from $G^y \backslash U_1$. Ben colors $(v_2, y)$ with any one of the two available colors, and Ann adds it to $U_2$. Now, Ann repeats the procedure starting from $(v_2, y)$, and continues to select a fixed vertex, that is a neighbor of a vertex in $U_2$, adds it to $U_2$, until there are no more vertices in $U_2$ that have a fixed neighbor in $G^y \backslash (U_1 \cup U_2)$. If $U_1 \cup U_2 \neq V(G^y)$, Ann selects an arbitrary vertex in $G^y \backslash (U_1 \cup U_2)$ and starts to build a new set $U_3$. When repeating this procedure, we produce pairwise disjoint sets $U_1, \ldots, U_k$. We claim that $\bigcup_{k=1}^n U_k = G^y$. (Note, that for each $i \in [k]$ the set $U_i$ contains exactly one vertex that was not fixed, namely the vertex that was added to $U_i$ first.). To finish the proof of this direction we have to show that the procedure described above is indeed Ann's winning strategy, that is, the sets $U_1, \ldots, U_k$ form a partition of $G^y$.

Suppose there is a step in Ann's strategy where she picks a vertex $(v, y)$ in $G^y$, and Ben cannot color it. This means that in the neighborhood of $(v, y)$ all colors from $\{1, 2, 3\}$ were used. Without loss of generality, let $(v, x)$ have color 1, and $(v_a, y), (v_b, y)$ be any two neighbors of $(v, y)$ with colors 2 and 3, respectively. If $(v_a, y) \in U_i$ and $(v_b, y) \in U_j$, for some $i < j$, then by Ann's strategy vertex $(v, y)$ would be colored before any vertex in $U_j$, because it would have been fixed. Hence, both vertices $(v_a, y)$ and $(v_b, y)$ must belong to the same set $U_i$ for some $i \in [k]$. As mentioned above, $U_i$ contains exactly one non-fixed vertex, say $(v_c, y)$. Note that $c \neq a$ and $c \neq b$, for otherwise $(v, y)$ would have been fixed and would be properly colored as one of the fixed neighbors of $(v_c, y)$. According to the construction of the $U$-sets, the subgraph $G[U_i]$ is connected. Hence, there exists a path $P'$ in $G[U_i]$ from vertex $(v_a, y)$ to vertex $(v_c, y)$ such that all vertices in $P'$, except for $(v_c, y)$, were fixed. Similarly, there is a path $P''$ in $G[U_i]$ from vertex $(v_b, y)$ to vertex $(v_c, y)$ such that all vertices in $P''$, except for $(v_c, y)$, were fixed. (Note that paths $P'$ and $P''$ might have more than one vertex, i.e. vertex $(v_c, y)$, in common.) Since the color of $(v_a, y)$ is 2, and because all vertices (except $(v_c, y)$) of $P'$ were fixed, there is a unique way how Ben assigned colors to them. The same must be true for the vertices of $P''$, since $(v_b, y)$ has color 3. Figure 1 shows the initial parts of the colorings of both paths.

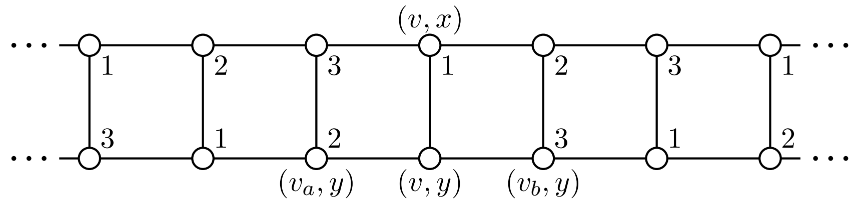


Figure 1: Proper colorings of paths $P'$ and $P''$

Suppose that vertex $(v_c, y)$ received color 1. According to the colors on the path $P'$, vertex $(v_c, x)$ must have received color 2, and according to the colors on the path $P''$, vertex $(v_c, x)$ must have been colored by 3, which is clearly a contradiction. Analogously, if $(v_c, y)$ received color 2, then $(v_c, x)$ must be simultaneously colored with 3 and 1, and if $(v_c, y)$ received color 3, then $(v_c, x)$ must be simultaneously colored with 1 and 2. In either of the three cases we get a contradiction to the fact that there is a step in Ann's strategy in which Ben cannot color a vertex.

We have shown that Ann's strategy produces an indicated coloring of $G \,\square\, K_2$ with three colors. Hence, $\chi_i(G \,\square\, K_2) \leq 3$, and since $G \,\square\, K_2$ is not bipartite, $\chi_i(G \,\square\, K_2) = 3$.

The converse is a special case of Lemma 2 for $H = K_2$ and $k = 3$, combined with the fact that the indicated chromatic number is 2 precisely for bipartite graphs. $\square$

**Theorem 4** *Let $G$ be a graph. Then $\chi_i(G) = 3$, if and only if $\chi_i(G \,\square\, K_{m,n}) = 3$ for $m, n \geq 1$.*

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_r\}$ and $V(K_{m,n}) = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$.

Suppose $\chi_i(G) = 3$. This means that Ann has a winning strategy in $G$ with three colors. Ann can use the same winning strategy in the fiber $G^{x_1}$ of $G$, which is isomorphic to $G$. Thus, $G^{x_1}$ can be properly colored with colors from $\{1, 2, 3\}$. The subgraph $G^{x_1} \,\square\, G^{y_1}$ is isomorphic to the graph $G \,\square\, K_2$, hence Ann has a winning strategy in the fiber $G^{y_1}$ using the same strategy as described in the proof of Theorem 3. Ann also uses the winning strategy described in the proof of Theorem 3 for the fibers $G^{y_2}, G^{y_3}, \ldots, G^{y_n}$ (note that there are no edges between fibers $G^{y_2}, G^{y_3}, \ldots, G^{y_n}$). After this step, all the fibers $G^{x_1}, G^{y_1}, G^{y_2}, \ldots, G^{y_n}$ are properly colored with three colors. In the remainder of the proof, Ann will use the following strategy.

First, if there is a fiber ${}^{v_i}K_{m,n}$, for some $i \in [r]$, in which there exist vertices $(v_i, y_l)$ and $(v_i, y_k)$, $l, k \in [n]$, which were colored with two different colors (note, that vertices $(v_i, y_1), (v_i, y_2), \ldots, (v_i, y_n)$ could have been colored with the same color, because there are no edges between them, or at most with two different colors, but they could not be colored with the color which was given to the vertex $(v_i, x_1)$, since this vertex is a neighbor of all vertices $(v_i, y_1), (v_i, y_2), \ldots, (v_i, y_n)$). In the next step Ann shows to Ben all vertices $(v_i, x_2), (v_i, x_3), \ldots, (v_i, x_m)$. These vertices have in their neighborhood two different colors (the colors of vertices $(v_i, y_l)$ and $(v_i, y_k)$), hence Ben must color them with the color with which the vertex $(v_i, x_1)$ is colored. Next, Ann repeats this procedure in all fibers ${}^{v_i}K_{m,n}$, $i \in [r]$, in which there exist vertices $(v_i, y_l)$ and $(v_i, y_k)$, $l, k \in [n]$, which were colored with two different colors. After this step, uncolored vertices exist only in the fibers ${}^{v_j}K_{m,n}$, $j \in [r]$, in which all vertices $(v_j, y_h)$, $h \in [n]$, have the same color. Therefore, two colors are available for them.

In the next moves, Ann will show to Ben uncolored vertices in the fibers $G^{x_2}, G^{x_3}, \dots, G^{x_m}$, in this order. She will use a similar procedure as in the first part of the proof of Theorem 3, namely, she will again construct partitions of those fibers into sets $U_i$. She starts with the fiber $G^{x_2}$. Let $U_1$ be the set of vertices in the fiber $G^{x_2}$, which are already colored. Let us prove that they do not have a neighbor with a fixed color. Let $(v_i, x_2)$ be an arbitrary vertex in $G^{x_2}$-fiber already colored and $(v_j, x_2)$ its uncolored neighbor in the same fiber.

We prove that $(v_j, x_2)$ is not a fixed vertex. Without loss of generality suppose that $(v_i, x_2)$ received color 1. The vertex $(v_i, x_2)$ was colored as a fixed vertex, otherwise it would not have been colored yet. This means there is some neighbor of vertex $(v_i, x_2)$, say $(v_i, y_l)$, for some $l \in [n]$, with color 2 and some neighbor, say $(v_i, y_k)$, for some $k \in [n]$, with color 3. Since the vertex $(v_j, x_2)$ is not yet colored, all its neighbors $(v_j, y_1), (v_j, y_2), \dots, (v_j, y_n)$ have the same color. But the color of these vertices is not 2, because the vertex $(v_i, y_l)$, which is a neighbor of the vertex $(v_j, y_l)$, has neither color 2 nor color 3, because the vertex $(v_i, y_k)$, which is a neighbor of the vertex $(v_j, y_k)$, has color 3. Therefore the vertices $(v_j, y_1), (v_j, y_2), \dots, (v_j, y_n)$ are colored with color 1. It follows that all neighbors of $(v_j, x_2)$ have color 1, thus a color is not fixed for $(v_j, x_2)$.

Since no neighbor of vertices in $U_1$ has a fixed color, Ann lets $U_2 = \emptyset$ and choses an arbitrary uncolored vertex and adds it to $U_2$. Then she picks all its neighbors with a fixed color and adds them to $U_2$. She continues to choose an arbitrary vertex of $U_2$, that still has some fixed neighbors in $G^{x_2} \backslash (U_1 \cup U_2)$ and adds them to $U_2$. Ann continues with this procedure until there are no more vertices in $U_2$ with a fixed neighbor in $G^{x_2} \backslash (U_1 \cup U_2)$. If $U_1 \cup U_2 = V(G)$, then all vertices of $G^{x_2}$ have been properly colored, otherwise $U_1 \cup U_2 \subsetneq G^{x_2}$, therefore Ann sets $U_3 = \emptyset$ and repeats the procedure starting from an arbitrary uncolored vertex of $G^{x_2} \backslash (U_1 \cup U_2)$. After some steps, we construct pairwise disjoint sets $U_1, U_2, \dots, U_k$ for which we claim that $\bigcup_{k=1}^{n} U_k = G^{x_2}$.

To finish the proof we have to show that with the procedure described above all vertices of the fiber $G^{x_2}$ have been properly colored. The proof is similar to the proof of Theorem 3. Indeed, each set $U_i$, for $i \in \{2, \dots, k\}$, contains exactly one vertex that was not fixed when it was colored, and this is the vertex that was added to $U_i$ first. In the same way as in the proof of Theorem 3 we find paths $P'$ and $P''$ within a fixed $U_i$ for some $i \in \{2, \dots, k\}$, that follow a fixed pattern. Note that for every vertex $(v_j, x_2)$ from $U_i$ all its neighbors $(v_j, y_1), (v_j, y_2), \dots, (v_j, y_n)$ have the same color, which is a similar situation as in the strategy for $G \,\square\, K_2$ in the proof of Theorem 3, where there is only one color in the first $G$-fiber which affects the coloring of each vertex in the second $G$-fiber.

Ann then repeats the same procedure, as described for the fiber $G^{x_2}$, for the fibers $G^{x_3}, \dots, G^{x_m}$ and produces a proper coloring of the whole graph $G \,\square\, K_{m,n}$ with three colors.

The converse again follows from Lemma 2. □

Given a graph $G$, $\omega(G)$ denotes the size of the largest clique in $G$.

**Lemma 5** *If $G$ is a block graph, then $\chi_i(G) = \chi(G) = \omega(G)$.*

**Proof.** Let $G$ be a block graph and $B_1, B_2, \dots, B_k$ its blocks, and let $\omega(G) = m$. Ann chooses a block $B_i$, for any $i \in [k]$, and shows to Ben all the vertices of $B_i$ in an arbitrary order. Since all vertices in $B_i$ are adjacent, Ben will have to use different colors for these vertices, but he will not use more than $m$ colors. Note that blocks of $G$ share cut-vertices. Hence there exists a block $B_j$ of $G$, $j \neq i$, which has one of its vertices already colored. Ann chooses such a block $B_j$ and shows to Ben all uncolored vertices of this block in an arbitrary order. He will again have to use different colors on those vertices. Ann can repeat this procedure by finding a new uncolored block of $G$ which has not been completely colored but has one vertex already colored from previous steps. It is clear that Ann's strategy will work since there exists only one path between any two blocks of $G$. In the end, all vertices will be colored by Ben, and he will be able to do this with $m$ colors. Thus, $\chi_i(G) \leq m$. Since $K_m$ is a subgraph of $G$, we have $\chi_i(G) = m = \omega(G)$. $\square$

**Theorem 6** *If $G$ is a block graph, then $\chi_i(G \,\square\, K_2) = \chi_i(G)$.*

**Proof.** Let $V(G) = \{v_1, \ldots, v_n\}$, $V(K_2) = \{x, y\}$, and $B_1, B_2, \ldots, B_k$, $k < n$, be the blocks of $G$.

Suppose that $\chi_i(G) = m$. By Lemma 5, Ann has a winning strategy for $G$, and since $G^x$ is isomorphic to $G$, she can use the same strategy for $G^x$. After $G^x$ is colored by $m$ colors, Ann chooses an arbitrary block, say $B_1$. Let $V(B_1) = \{v_1, \ldots, v_l\}$, and note that $l \leq m = \omega(G)$. Ann's next move is picking an arbitrary vertex $(v_i, y)$, $v_i \in V(B_1)$, and Ben colors it with a color $c \in [m]$ which is different to the color of $(v_i, x)$.

If $l < m$, than Ann can pick all the remaining vertices of clique $B_1^y$ in any order. Ben colors them with $l < m$ different colors from the set $[m]$.

On the other hand, if $l = m$, then the clique $B_1 \times \{x\}$ contains a vertex $(v_j, x)$, $j \neq i$, which is colored with color $c$. The strategy of Ann is to pick all the vertices of the clique $B_1 \times \{y\}$, except the vertex $(v_j, y)$, in an arbitrary order. When all the vertices of the clique $B_1^y$, except the vertex $(v_j, y)$, have been colored, Ann picks the vertex $(v_j, y)$. With this she ensures that Ben will be able to color all vertices of $B_1 \times \{y\}$ including the vertex she picked last; notably, $(v_j, y)$ has $m+1$ neighbors, two of which, $(v_j, x)$ and $(v_i, y)$, are colored with the same color $c$. Hence, there is still one color available for the last vertex of the clique $B_1 \times \{y\}$.

In the next step, Ann chooses an arbitrary clique $B_i \times \{y\}$, for some $i \in \{2, \ldots, k\}$, which has at least one of its vertices colored (i.e., for $B_i$ which shares a vertex with $B_1$). Ann uses the strategy she used for $B_1^y$ on the block $B_i^y$ (keeping the vertex of $B_i^y$ having the same color as the vertex in $V(B_1^y) \cap V(B_i^y)$ to be colored as the last vertex of $V(B_i^y)$). By repeating this procedure, choosing blocks of $G^y$ that have a non-empty intersection with one of the blocks of $G^y$ that have already been colored, all vertices of $G^y$ will eventually be colored using $m$ colors. Thus, $\chi_i(G \,\square\, K_2) \leq m$, and since $m = \chi(G \,\square\, K_2) \leq \chi_i(G \,\square\, K_2)$, the proof is complete. $\square$

Theorem 6 can be extended as follows.

**Corollary 7** *Let $G$ be a block graph and $T$ be a tree. Then $\chi_i(G \,\square\, T) = \chi_i(G)$.*

**Proof.** Let $\{v_0, v_1, \ldots, v_n\}$ be the set of vertices of a tree $T$ ordered with the BFS algorithm, and let $v_0$ denote the root of $T$. Suppose that $\chi_i(G) = m$. By Lemma 5, Ann has a winning strategy for $G$ with $m$ colors, and since $G^{v_0}$ is isomorphic to $G$, she can use the same strategy for $G^{v_0}$. In the next step, Ann chooses the vertices of the fiber $G^{v_1}$ by using the same strategy as described in the proof of Theorem 6. The reason why she can use this strategy is that the subgraph $G^{v_0} \square G^{v_1}$ of $G \square T$ is isomorphic to $G \square K_2$. After the vertices of $G^{v_1}$ have been colored by Ben, she chooses the fibers $G^{v_2}, \dots, G^{v_n}$ in the BFS order, and repeats her strategy from $G^{v_1}$. In the end, a proper coloring of $G \square T$ using $m$ colors is obtained. $\square$

Finally, we consider the indicated chromatic number of the Cartesian product of two cycles, where one cycle is $C_3$, and show that $\chi_i(C_3 \square C_n) = 3$ for all integers $n \geq 3$.

**Proposition 8** *Let $C_n$ be a cycle of length $n \geq 3$. Then $\chi_i(C_3 \square C_n) = 3$.*

**Proof.** Let $V(C_3) = \{x_1, x_2, x_3\}$ and $V(C_n) = \{y_1, \dots, y_n\}$. It is clear, that the indicated chromatic number of the cycle is 3. Therefore Ann has a winning strategy in the fiber $C_3^{y_1}$, so she uses it. For the same reason, Ann has a winning strategy in the fiber ${}^{x_1}C_n$, and she uses it. Now the first row and the first column of the product $C_3 \square C_n$ are colored. In the remainder, Ann will use the following strategy, defined in the rows of the product: she will handle the rows $C_3^{y_2}$, $C_3^{y_3}$,...,$C_3^{y_n}$ in the natural order. In each row $C_3^{y_i}$, for $i \in \{2, 3, \dots, n\}$, she will pick the vertex $(x_2, y_i)$ before the vertex $(x_3, y_i)$ if the color of $(x_2, y_i)$ is fixed, and the vertex $(x_3, y_i)$ before the vertex $(x_2, y_i)$ if the color of $(x_3, y_i)$ is fixed. Note, that exactly one of these two vertices is always fixed. In this way, she will reach a fixed permutation of the colors from the set $\{1, 2, 3\}$ in all rows. Without loss of generality we can suppose that the vertex $(x_1, y_1)$ is colored with color 1, $(x_2, y_1)$ with color 2 and $(x_3, y_1)$ with color 3 as shown in the Figure 2.

Using the described strategy, all vertices of the graph $C_3 \square C_n$ will be properly colored with three colors. $\square$

We believe that the indicated chromatic number of the product of two cycles of arbitrary lengths is at most three. In the concluding remarks, we present a strategy of Ann, for which we think that it is a winning strategy.

# 3 Concluding remarks

In the previous section we proved that $\chi_i(C_3 \square C_n) = 3$, where $n \geq 3$. What follows is a strategy for Ann to color the product $C_n \square C_m$, $n, m \geq 3$, where at least one of $n$ or $m$ is odd. We strongly believe that the below described strategy is a winning strategy for Ann. Unfortunately, we were not able to prove it.

Let $V(C_n) = \{x_1, \dots, x_n\}$ and $V(C_m) = \{y_1, \dots, y_m\}$. As the indicated chromatic number of $C_n$ is at most 3, Ann has a winning strategy in the fiber $C_n^{y_1}$, which she applies. From the same reason, Ann has a winning strategy in the fiber ${}^{x_1}C_m$, which she applies next. After this, the first row and the first column of the product $C_n \square C_m$ are colored.

Ann's strategy will be defined in the rows of the product. We will say that Ann *handles a row* from left to right and afterwards from right to left if she shows to Ben only the uncolored fixed vertices from this row in the following way. Suppose that she chooses the fiber $C_n^{y_k}$, for some $k \in \{2, \dots, m\}$. Let $i \in [n]$ be the largest index such that $(x_1, y_k), \dots, (x_i, y_k)$ were all fixed uncolored vertices when Ann was picking them consecutively (Ann picks vertices from

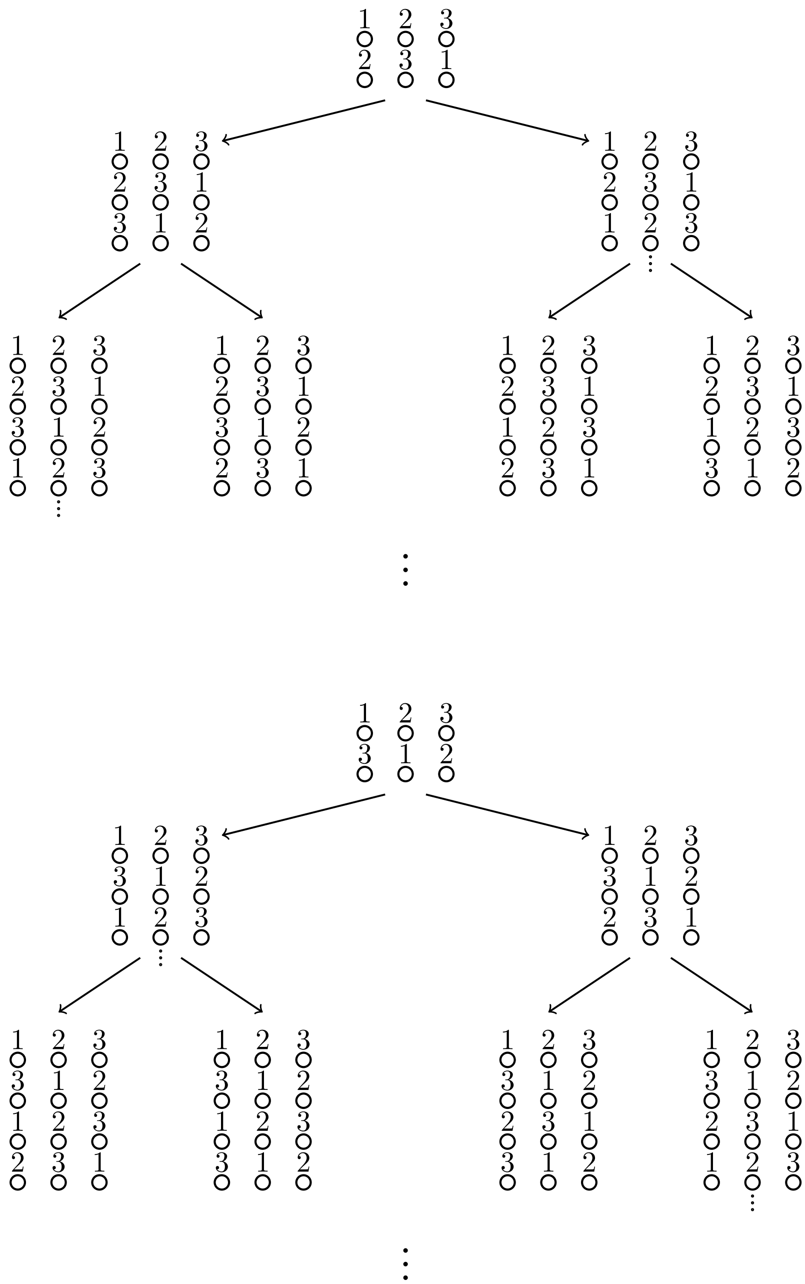

Figure 2: Proper colorings of product $C_3 \square C_n$

left to right in this row while they are fixed). If $i \neq n$, then let $j \in \{2, \ldots, n\}$ be the smallest index, if it exists, such that $(x_n, y_k), \ldots, (x_j, y_k)$ were all fixed uncolored vertices when Ann was picking them consecutively (Ann picks vertices from right to left in this row while they are fixed). Note that $i + 1 \leq j$. If $j = i + 1$, then all vertices were colored in this row (in this case it can be easily seen why the last vertex in this row can be colored). But if $j > i + 1$, then the vertices $(x_{i+1}, y_k), \ldots, (x_{j-1}, y_k)$ stay uncolored. In the case of $i = n$, we must check that Ben really can color the last vertex in the last fiber $C_n^{y_k}$, since this vertex now has tree colored vertices in its neighborhood.

Without loss of generality suppose that $(x_1, y_{k-1})$ was colored with color 1 and $(x_1, y_k)$ with color 2. Then $(x_2, y_{k-1})$ must have color 3 and $(x_2, y_k)$ fixed color 1.

1. If $n \equiv 0, 3 \pmod 6$, then the vertices $(x_1, y_k)$ and $(x_n, y_{k-1})$ have color 2, $(x_{n-1}, y_k)$ color 1, hence Ben can color $(x_n, y_k)$ with color 3.
2. If $n \equiv 1 \pmod 6$, then the vertices $(x_{n-1}, y_k)$ and $(x_n, y_{k-1})$ have color 3, $(x_1, y_k)$ color 2, hence Ben can color $(x_n, y_k)$ with color 1.
3. If $n \equiv 2, 4 \pmod 6$, then the vertices $(x_{n-1}, y_k)$ and $(x_n, y_{k-1})$ have color 3, $(x_1, y_k)$ color 2, hence Ben can color $(x_n, y_k)$ with color 1.
4. If $n \equiv 5 \pmod 6$, then the vertices $(x_1, y_k)$ and $(x_{n-1}, y_k)$ have color 2, $(x_n, y_{k-1})$ color 3, hence Ben can color $(x_n, y_k)$ with color 1.

The coloring of the graph $C_n \,\Box\, C_m$ can be done in several steps, in which a partition of $C_n \,\Box\, C_m$ into sets $U_i$ is built. In the beginning, let $U_1$ contain the vertices of the first row and the first column, which are already colored. Now, Ann handles the rows $C_n^{y_2}$, $C_n^{y_3}$,..., $C_n^{y_m}$ in this order, and adds the vertices, which were colored in this step, to $U_1$. After Ann handled the last row $C_n^{y_m}$, there could be some new fixed uncolored vertices in the row $C_n^{y_{m-1}}$, which were not fixed before. Therefore Ann handles again the row $C_n^{y_{m-1}}$ and adds newly colored vertices to $U_1$. But after $C_n^{y_{m-1}}$ is handled, again there could be some new fixed uncolored vertices in the row $C_n^{y_{m-2}}$, so Ann handles again this row. It is obvious that Ann must again handle the rows $C_n^{y_{m-1}}$, ...,$C_n^{y_2}$ in this order (some of those fibers might have already been colored before, so she just skips those rows), and add all newly colored vertices to $U_1$. We call this procedure, when Ann handles the rows from the top to the bottom and afterwards from the bottom to the top, the *backtracking* procedure. Ann continuous to apply the backtracking procedure until either the whole graph $C_n \,\Box\, C_m$ has been colored or until there are no more uncolored fixed vertices available. She adds all vertices colored during that process to $U_1$.

If $U_1 = V(C_n \,\Box\, C_m)$, then all vertices of $C_n \,\Box\, C_m$ have been colored. Otherwise $U_1 \subsetneq V(C_n \,\Box\, C_m)$. Let $U_2 = \emptyset$. Ann now determines the first row in which there is some uncolored vertex and shows to Ben the uncolored vertex with the smallest second index, and repeatedly applies the backtracking procedure on the graph $C_n \,\Box\, C_m$ starting from this row/vertex. She adds newly colored vertices to $U_2$. The procedure again ends if either the whole graph $C_n \,\Box\, C_m$ has been colored or there are no more uncolored vertices with a fixed color. If $U_1 \cup U_2 \neq V(C_n \,\Box\, C_m)$, Ann again selects the first row in which there is some uncolored vertex, and repeats the backtracking procedure. When repeating this process, we produce pairwise disjoint sets $U_1, \ldots, U_k$, for which $\bigcup_{k=1}^{n} U_k = V(C_n \,\Box\, C_m)$. However, one needs to show that this is indeed always possible to achieve, and that we consequently obtain a proper coloring of the whole graph.

We think that this strategy enables Ann to win the game on the Cartesian product of two cycles with only 3 available colors, which would give $\chi_i(C_n \,\Box\, C_m) = 3$ for all integers $m$ and $n$, where at least one of them is odd. Moreover, we pose the following, much more general question, for which we suspect it has a positive answer.

**Question 1** *Is it true that* $\chi_i(G \,\Box\, H) = \max\{\chi_i(G), \chi_i(H)\}$ *holds for all graphs* $G$ *and* $H$*?*

## Acknowledgements

B.B. and M.J. acknowledge the financial support from the Slovenian Research Agency (research core funding No. P1-0297 and research project J1-9109).

## References

[1] S.D. Andres, The incidence game chromatic number, Discrete Appl. Math. 157 (2009) 1980–1987.

[2] T. Bartnicki, B. Brešar, J. Grytczuk, M. Kovše, Z. Miechowicz, I. Peterin, Game chromatic number of Cartesian product graphs, Electron. J. Combin. 15 (2008) #R72, 13 pp.

[3] T. Bartnicki, J. Grytczuk, H.A. Kierstead, The game of arboricity, Discrete Math. 308 (2008) 1388–1393.

[4] T. Bartnicki, J. Grytczuk, H. A. Kierstead, X. Zhu, The map coloring game, Amer. Math. Monthly 144 (2007) 793–803.

[5] H.L. Bodlaender, On the complexity of some coloring games, Internat. J. Found. Comput. Sci. 2 (1991) 133–147.

[6] B. Bosek, J. Grytczuk, G. Jakóbczak, Majority coloring game, Discrete Appl. Math. 255 (2019), 15–20.

[7] C. Charpentier, E. Sopena, Incidence coloring game and arboricity of graphs, Lecture Notes Comp. Sci. 8288 (2013) 106–114.

[8] T. Dinski, X. Zhu, Game chromatic number of graphs, Discrete Math. 196 (1999) 109–115.

[9] A.S. Fraenkel, Combinatorial games: selected bibliography with a succinct gourmet introduction, Electron. J. Combin., Dynamic Survey 2 (2007) 78pp; `https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS2`.

[10] P. Francis, S.F. Raj, Indicated coloring of Cartesian product of graphs, Lecture Notes Comp. Sci. 10398 (2017) 63–68.

[11] M. Gardner, Mathematical games, Scientific American 244 (1981) 18–26.

[12] A. Grzesik, Indicated coloring of graphs, Discrete Math. 312 (2012) 3467–3472.

[13] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs, Second Edition*, CRC Press, Boca Raton, FL, 2011.

[14] H.A. Kierstead, A. Kostochka, Efficient graph packing via game coloring, Combin. Probab. Comput. 18 (2009) 765-774.

[15] H.A. Kierstead, T. Trotter, Competitive colorings of oriented graphs, Electron. J. Combin. 8 (2001) #R12, 15pp.

[16] M. Lasoń, Indicated coloring of matroids, Discrete Appl. Math. 179 (2014) 241–243.

[17] J. Nešetřil, E. Sopena, On the oriented game chromatic number, Electron. J. Combin. 8 (2001) R#14, 13pp.

[18] R.P. Raj, S.F. Raj, H.P. Patil, On indicated coloring of graphs, Graphs Combin. 31 (2015) 2357–2367.

[19] S.F. Raj, R.P. Raj, H.P. Patil, On indicated chromatic number of graphs, Graphs Combin. 33 (2017) 203–219.

[20] G. Sabidussi, Graphs with given group and given graph-theoretical properties, Canad. J. Math. 9 (1957) 515–525.

[21] Zs. Tuza, X. Zhu, Colouring games, in [Topics in chromatic graph theory, *Cambridge Univ. Press*, Cambridge, 2016], 304–326.

[22] X. Zhu, Game coloring the Cartesian product of graphs, J. Graph Theory 59 (2008) 261–278.