

The k -path vertex cover: general bounds and chordal graphs

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Abstract

For an integer $k \geq 3$, a k -path vertex cover of a graph $G = (V, E)$ is a set $T \subseteq V$ that shares a vertex with every path subgraph of order k in G . The minimum cardinality of a k -path vertex cover is denoted by $\psi_k(G)$. We give estimates — mostly upper bounds — on $\psi_k(G)$ in terms of various parameters, including vertex degrees and the number of vertices and edges. The problem is also considered on chordal graphs and planar graphs.

Key words: transversal, k -path vertex cover, chordal graph.

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1 Introduction

A k -path vertex cover (or P_k -transversal) in a graph G is a set $T \subseteq V(G)$ that shares at least one vertex with every path of order k in G ; i.e., $G - T$ does not contain any P_k subgraph. In this paper we study the minimum cardinality of a k -path vertex cover in G , denoted by $\psi_k(G)$. Remark that the case of $k = 2$ means just the vertex cover (transversal) that is the complement of an independent set. It immediately follows that $\psi_2(G) = |V(G)| - \alpha(G)$, where $\alpha(G)$ denotes the size of a maximum stable set and is called the *independence number* of G . Since these are central notions in graph theory and the literature contains many related deep results, here we concentrate on the cases with $k \geq 3$. The case of $k = 3$ is in close relation to the so-called dissociation number, which was firstly studied by Yannakakis [26]. A subset of vertices in a graph G is called a dissociation set if it induces a subgraph with maximum degree at most 1. The size of a maximum dissociation set in G is called the *dissociation number* of G and is denoted by $\text{diss}(G)$. Thus, $\psi_3(G) = |V(G)| - \text{diss}(G)$. Several other connections can be found; for instance in [3], k -path vertex cover was shown to be related to k -path partition.

Application in Information Technology. The motivation for the invariant $\psi_k(G)$, which was introduced in [5], arises from communications in wireless sensor networks, where the data integrity is ensured by using Novotný's k -generalized Canvas scheme [19]. The topology of wireless sensor networks can be represented by a graph, in which vertices represent sensor devices and edges represent communication channels between them. We suppose that there are protected and unprotected sensor devices in the model. The attacker is unable to copy secrets from a protected device; however, an unprotected device can be captured by the attacker who can gain control over it. During the deployment and initialization of a sensor network, it should be ensured that at least one protected vertex exists on each path of order k in the communication graph. The placement of protected sensors in a network is usually expensive. Thus an important goal is to minimize the cost of the network by minimizing the number of protected vertices, which coincides with the calculation of a minimum k -path vertex cover in a graph that represents such a network.

Known results on algorithmic complexity. It was proved in the pilot paper that the computation of $\psi_k(G)$ is in general an NP-hard problem, but was shown to be solvable in linear time over the class of trees.

The k -path vertex cover problem has gained much attention in the area of computational graph theory. Approximation algorithms for $\psi_3(G)$ were given by Tu et al. in [24, 23] and an exact algorithm for computing $\psi_3(G)$ in running time¹ $O^*(1.5171^n)$ for a graph of order n was presented in [12], and was later improved to $O^*(1.366^n)$ in [25]. The parametrized version of the same problem was also considered. The goal

¹suppressing polynomially bounded factors by the O^* -notation

is to decide whether there is a k -path vertex cover in G of size at most ℓ , where k and ℓ are fixed positive integers. A fixed parameter tractable (FPT) algorithm for the 3-path vertex cover whose time complexity is $O^*(1.713^\ell)$ was given by Tsur in [22]. Tsur recently presented a FPT algorithm for the 4-path vertex cover which runs in $O^*(2.619^\ell)$ [21].

The weighted version of k -path vertex cover was introduced by Brešar et al. in [6]. In this version vertices are given weights and the problem is to find a minimum-weight set such that the graph obtained by deleting this set of vertices has no path P_k as a subgraph. Some special classes of graphs were considered, for instance, complete graphs and cycles, and an algorithm that computes the weighted k -path vertex cover number of a tree with time complexity $O(k \cdot |V(G)|)$ was presented. Approximation algorithms for the weighted and connected versions of the k -path vertex cover were considered in [15], and later also in [20] where the emphasis was primarily on $k = 3$. In both cases an extra assumption was present, namely that the subgraph induced by a k -path vertex cover was connected. The connected version of the k -path vertex cover was also studied in [7, 17].

Known results on graph products. The k -path vertex cover problem was heavily studied on several types of graphs products. Already in [4] the Cartesian product of two paths, i.e. grid graphs, was considered. In [16] those results were extended to the Cartesian product of three paths. The results on grid graphs were later improved in [11], and the ideas of the proofs were used on the strong product of paths. In the same paper an upper and a lower bound were given for the k -path vertex cover of the lexicographic product of arbitrary graphs. The upper bound was shown to be tight for any choice of factor graphs when $k = 3$. The k -path vertex cover of the rooted product of arbitrary graphs was considered in [10].

Some known lower and upper bounds. Throughout many articles on the k -path vertex cover the following problem appeared to gain much attention.

Problem 1 *For a given k , determine the set of pairs (a, b) of nonnegative reals such that $\psi_k(G) \leq an + bm$ holds for every graph with n vertices and m edges. Study the same also for interesting classes of graphs.*

In the next theorems we cite some tight related results from the paper of Brešar et al. [5], which will be applied later.

Theorem 2 ([5]) *Let T be a tree of order n and k an integer with $k \geq 3$. Then, we have $\psi_k(T) \leq n/k$.*

Theorem 3 ([5]) *Let G be a graph of order n and size m . Then*

$$\psi_3(G) \leq \frac{2n + m}{6} \quad \text{and} \quad \psi_3(G) \leq \frac{m}{2}.$$

Moreover, if G is subcubic, then $\psi_3(G) \leq n/2$ also holds.

Note that $\psi_3(G) \leq m/2$ follows from the procedure in which we sequentially remove a vertex of degree at least 2 from G and put it into the 3-path vertex cover. During this procedure, we delete at most $m/2$ vertices and what remains at the end is a graph with maximum degree at most 1, hence P_3 -free. Further, if $\Delta(G) \geq 3$, then the strict inequality $\psi_3(G) < m/2$ holds as we may remove a vertex and at least three incident edges in the first step of the procedure.

The relation $\psi_3(G) \leq n/2$ can also be obtained via an iterated algorithmic process. Here, starting with any partition (V_1, V_2) of the vertex set of a subcubic graph, a vertex which has more than one neighbor in its partition class is moved to the other class, and hence increasing the size of the cut, until no such vertices remain. Then each of V_1 and V_2 is a 3-path vertex cover.

In [4] the upper bound

$$\psi_3(G) \leq \frac{\ell}{\ell + 2}n + \frac{1}{(\ell + 1)(\ell + 2)}m$$

is established, where $\ell = \lceil \frac{m}{n} \rceil - 1$. Also this bound is tight.

Our results. In this paper we continue this track of research and prove upper bounds on $\psi_k(G)$ in terms of various parameters, including vertex degrees and the number of vertices and edges. We generalize the result of [4] for d -regular graphs,

$$\psi_k(G) \geq \frac{d - k + 2}{2d - k + 2}n, \tag{1}$$

to arbitrary graphs in terms of minimum and maximum degree. We also consider the k -path vertex cover problem on some famous classes of graphs, such as chordal graphs and planar graphs. Our paper is structured as follows. In Section 2 we study Problem 1 and prove general upper bounds on $\psi_3(G)$, $\psi_4(G)$, and $\psi_k(G)$ in terms of the order and size of G .

We also present a method that can be applied to generate, from feasible pairs (a, b) of Problem 1, further feasible pairs (a', b') . In Section 3 we give lower and upper bounds on $\psi_k(G)$ with functions of vertex degrees. Chordal graphs are considered in Section 4. Section 5 describes a way to improve the estimates further, and contains more open problems. In particular, Subsection 5.2 discusses the problem on planar graphs. Some further problems are mentioned in the concluding section.

Notation

We use standard notation as follows. For a simple undirected graph G we usually denote the order $|V(G)|$ and the size $|E(G)|$ by n and m . The notations $\delta(G)$, $\Delta(G)$, $\chi(G)$, and $\omega(G)$ stand for the *minimum vertex degree*, the *maximum vertex degree*, the *chromatic number*, and the *clique number* of G , respectively. We will simply write δ , Δ , χ , and ω if G is clear from the context. The *degree* of a vertex $v \in V(G)$ in graph

G is denoted by $d_G(v)$, and the set of its neighbors is denoted by $N_G(v)$ and called the *open neighborhood* of v . Moreover, we define $N_G[v] = N_G(v) \cup \{v\}$ as the *closed neighborhood* of v . Also here, if G is understood, we will simply write $d(v)$, $N(v)$ and $N[v]$. The average vertex degree is $\bar{d} = \bar{d}(G) = 2m/n$. The subgraph induced by vertex set $Y \subseteq V(G)$ in G is referred to as $G[Y]$. Notations P_k , C_k , and K_k stand for the path, cycle, and complete graph of order k , respectively.

2 Upper bounds related to Problem 1

Here our goal is to make a modest step in the direction of solving Problem 1. Suppose that $an + bm$ is a general upper bound on $\psi_k(G)$ for an integer $k \geq 3$. If $b = 0$, the best estimation is obtained with $a = 1$, as shown by complete graphs which have $\psi_k(K_n) = n - k + 1$; and for $a = 0$ with $k = 3$ the best pair is $b = 1/2$. The corresponding upper bound is attained by graphs in which every component is a P_3 . Also, $(a, b) = (1/3, 1/6)$ provides a valid upper bound, by Theorem 3.

Theorem 4 *If G is a graph of order n and size m , then*

$$\psi_3(G) \leq \frac{n + m}{4},$$

where equality holds if and only if all components of G are isomorphic to C_4 .

Proof. We proceed by induction on n and suppose that G is given on $n \geq 3$ vertices. If $\Delta(G) \leq 3$, the inequalities $\psi_3(G) \leq n/2$ and $\psi_3(G) \leq m/2$ from Theorem 3 imply the statement. Otherwise, there exists a vertex $v \in V(G)$ with $d_G(v) \geq 4$. By the induction hypothesis, $G' = G - v$ satisfies the inequality. Note that G' has $n' = n - 1$ vertices and $m' \leq m - 4$ edges and further, if T' is a 3-path vertex cover in G' , then $T' \cup \{v\}$ is a 3-path vertex cover in G . We may conclude that

$$\psi_3(G) \leq \psi_3(G') + 1 \leq \frac{n' + m'}{4} + 1 \leq \frac{n + m - 5}{4} + 1 < \frac{n + m}{4}.$$

This completes the proof for the upper bound.

Now, we consider the graphs attaining the upper bound in the theorem. Let G be any graph with $\psi_3(G) = \frac{n+m}{4}$. It suffices to prove that if G is connected, then $G \cong C_4$. It follows from Theorem 3 that $n = m$, for otherwise $\psi_3(G) \leq \min\{\frac{n}{2}, \frac{m}{2}\} < \frac{n+m}{4}$. As we discussed after Theorem 3, $\Delta(G) \geq 3$ would imply $\psi_3(G) < m/2$ and, therefore, $\psi_3(G) < \frac{n+m}{4}$ which is a contradiction. Hence, G satisfies $\Delta(G) \leq 2$ and $n = m$. This implies $G \cong C_n$. Then $\psi_3(G) = \lceil \frac{n}{3} \rceil$, which is easily seen to be smaller than $n/2$ unless $n = 4$. \square

For the upper bound stated in Theorem 5 below, we also have a sharp example, namely $H = K_6 - \mathcal{M}$ which is obtained from the complete graph K_6 by removing a perfect matching \mathcal{M} . Then we have $n = 6$, $m = 12$, and $\psi_3(H) = 4$.

Theorem 5 *Let G be a graph of order n and size m . Then*

$$\psi_3(G) \leq \frac{4n + m}{9}.$$

Proof. By Theorem 3, every graph satisfies $\psi_3(G) \leq \frac{2n+m}{6}$. To show that $\psi_3(G) \leq \frac{4n+m}{9}$ is also true, we first note that the inequality clearly holds for graphs of order at most 2. Then, we proceed by induction on the order n of G and consider two cases.

- If $m \leq 2n$, the inequality chain

$$\frac{4n + m}{9} = \frac{n}{3} + \frac{m}{6} + \left(\frac{n}{9} - \frac{m}{18}\right) \geq \frac{n}{3} + \frac{m}{6} \geq \psi_3(G)$$

establishes the statement without using the induction hypothesis.

- If $m > 2n$, then $\Delta(G) \geq 5$ follows. Let v be a vertex of degree at least 5. Deleting v from G , we obtain the graph G' which is of order $n' = n - 1$ and of size $m' \leq m - 5$. By the induction hypothesis, G' satisfies the inequality, i.e. $\psi_3(G') \leq (4n' + m')/9$. Moreover, if T' is a 3-path vertex cover in G' , then $T' \cup \{v\}$ covers all P_3 subgraphs in G . This yields $\psi_3(G) \leq \psi_3(G') + 1$. Then, we have the desired result by the following relations:

$$\psi_3(G) \leq \psi_3(G') + 1 \leq \frac{4n' + m'}{9} + 1 \leq \frac{4(n-1) + (m-5)}{9} + 1 = \frac{4n + m}{9}.$$

This completes the proof. □

We also present a general result that shows how one can obtain new feasible pairs (a', b') if a feasible pair (a, b) is already given.

Theorem 6 *Let a and b be nonnegative reals and k an integer with $k \geq 3$ such that $\psi_k(G) \leq an + bm$ holds for every graph G . Moreover, let x be an arbitrary positive integer which satisfies*

$$\frac{1 - a - b}{b} \leq x \leq \frac{2 - 2a}{b}, \tag{2}$$

and define a' and b' as follows:

$$a' = a + bx - \frac{x(1 + b - a)}{x + 2}, \quad b' = \frac{2 - 2a - bx}{x + 2}.$$

Then, the inequality

$$\psi_k(G) \leq a'n + b'm \tag{3}$$

holds for every graph G .

Proof. Let a, b, a', b' and x satisfy the conditions in the theorem. We introduce the notation

$$y = \frac{x(a + b + bx - 1)}{x + 2}.$$

By (2), we have $a + b + bx - 1 \geq 0$ that implies $y \geq 0$. One can check that $a' = a + y$ and $b' = b - \frac{2y}{x}$. As the second inequality in (2) is equivalent to $bx \geq 2y$, we have $b' \geq 0$ and $a' \geq a \geq 0$. Consequently, the right-hand side of (3) is always nonnegative and inequality (3) clearly holds for small graphs with $n \leq k - 1$. We proceed by induction on n . Consider a graph G of order n and size m .

- First, suppose that $m \leq \frac{xn}{2}$. This implies $n - \frac{2m}{x} \geq 0$. Using this relation and rewriting the formula $a'n + b'm$, we get

$$a'n + b'm = (a + y)n + \left(b - \frac{2y}{x}\right)m = (an + bm) + y\left(n - \frac{2m}{x}\right) \geq an + bm \geq \psi_k(G)$$

that verifies (3) for the first case.

- Next, suppose that the strict inequality $m > \frac{xn}{2}$ holds. It implies $\bar{d}(G) > x$ and therefore $\Delta(G) \geq x + 1$ holds. Let v be a vertex with $d_G(v) \geq x + 1$. Deleting v from G , we obtain a graph G' with $n' = n - 1$ vertices and $m' \leq m - (x + 1)$ edges. Since every edge that was deleted from G is covered by v , we have $\psi_k(G) \leq \psi_k(G') + 1$. By our hypothesis, G' satisfies (3). The following computation proves the same property for G :

$$\begin{aligned} a'n + b'm &\geq (a'n' + b'm') + a' + b'(x + 1) \\ &\geq \psi_k(G') + (a + y) + \left(b - \frac{2y}{x}\right)(x + 1) \\ &= \psi_k(G') + a + bx + b - \frac{x + 2}{x}y \\ &= \psi_k(G') + a + bx + b - (a + b + bx - 1) \\ &= \psi_k(G') + 1 \geq \psi_k(G). \end{aligned}$$

This finishes the proof of the theorem. □

Since the formulation of the above theorem is quite technical, we present some upper bounds which can be obtained starting with Theorem 5 and applying Theorem 6 iteratively. We begin with the case of $k = 3$ and with the upper bound $\frac{4}{9}n + \frac{1}{9}m$ and set $a_0 = 4/9$, $b_0 = 1/9$. Then for each $1 \leq i \leq 10$, we apply Theorem 6 with $x = i + 4$, $a = a_{i-1}$, and $b = b_{i-1}$.

Corollary 7 For every graph G of order n and size m , the following inequalities hold:

$$\begin{aligned}\psi_3(G) &\leq \frac{11}{21}n + \frac{5}{63}m, \\ \psi_3(G) &\leq \frac{7}{12}n + \frac{5}{84}m, \\ \psi_3(G) &\leq \frac{17}{27}n + \frac{5}{108}m, \\ \psi_3(G) &\leq \frac{2}{3}n + \frac{1}{27}m, \\ \psi_3(G) &\leq \frac{23}{33}n + \frac{1}{33}m, \\ \psi_3(G) &\leq \frac{13}{18}n + \frac{5}{198}m, \\ \psi_3(G) &\leq \frac{29}{39}n + \frac{5}{234}m, \\ \psi_3(G) &\leq \frac{16}{21}n + \frac{5}{273}m, \\ \psi_3(G) &\leq \frac{7}{9}n + \frac{1}{63}m, \\ \psi_3(G) &\leq \frac{19}{24}n + \frac{1}{72}m.\end{aligned}$$

Proposition 8 If G is a graph of maximum degree $\Delta(G) \leq 2$, then

$$\psi_k(G) \leq \frac{2}{k+1}n \quad \text{and} \quad \psi_k(G) \leq \frac{2}{k+1}m$$

hold for every $k \geq 3$. Further, the first inequality is tight if and only if each component of G is a $(k+1)$ -cycle; the second one is tight if and only if each component is either an isolated vertex or a $(k+1)$ -cycle.

Proof. It follows from $\Delta(G) \leq 2$ that every component of G is either a path or a cycle. Since ψ_k is an additive invariant and

$$\psi_k(P_n) = \left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{m(P_n) + 1}{k} \right\rfloor \leq \psi_k(C_n) = \left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{m(C_n)}{k} \right\rceil$$

holds for every $n \geq k$, moreover $\psi_k(P_n) = \psi_k(C_n) = 0$ holds if $n < k$, we easily derive

$$\psi_k(G) \leq \frac{2}{k+1}m \leq \frac{2}{k+1}n$$

and identify the sharp cases. □

Theorem 9 Let G be a graph of order n and size m . Then

$$\psi_4(G) \leq \frac{n + 3m}{10}.$$

Proof. We proceed by induction on n . If $\Delta(G) \leq 2$, then $5\psi_4(G) \leq 2n$ and $15\psi_4 \leq 6m$ follow by Proposition 8. Therefore, we have $20\psi_4(G) \leq 2n + 6m$ and the statement follows. If $\Delta(G) \geq 3$, there is a vertex v of degree at least 3 and we may consider the graph $G' = G - v$. Since G' has $n' = n - 1$ vertices, $m' \leq m - 3$ edges and, by our hypothesis, it satisfies $\psi_4(G') \leq (n' + 3m')/10$,

$$\psi_4(G) \leq \psi_4(G') + 1 \leq \frac{n' + 3m'}{10} + 1 \leq \frac{(n-1) + 3(m-3)}{10} + 1 = \frac{n + 3m}{10}$$

is obtained that completes the proof. \square

Theorem 9 holds with equality if each component of G is a C_5 . (Currently the 5-cycle is the only connected graph for which we know the equality.) Moreover, using Theorem 6, further upper bounds on ψ_4 can be generated. However, we do not list such bounds here. Instead, generalizing Theorem 6 we close this section with describing a general class of recursions showing how further feasible pairs (a', b') can be obtained from an already known feasible pair (a, b) .

Theorem 10 *Assume that $\psi_k(G) \leq an + bm$ is a universally valid inequality for all graphs G . Suppose further that the real numbers $q, w > 0$ and the integer $x > 0$ satisfy the following inequalities:*

$$w \leq 2q, \quad a + qx < 1, \quad w \leq \frac{(a + b - 1) + (q + b)x}{x + 1}.$$

Then $\psi_k(G) \leq a'n + b'm$ also holds for every graph G with the values

$$a' = a + qx, \quad b' = b - w.$$

Proof. Let us note first that under the given conditions we have $b' > 0$. Indeed, from the assumed inequalities we obtain

$$1 \leq a + qx + bx + b - (x + 1)w < 1 + (b - w)(1 + x).$$

If G has average degree at most x , then $m \leq xn/2$ holds, thus

$$\begin{aligned} \psi_k(G) &\leq an + bm = (a' - qx)n + (b' + w)m = a'n + b'm - (qxn - wm) \\ &\leq a'n + b'm - (qxn - 2qm) \leq a'n + b'm. \end{aligned}$$

Otherwise there exists a vertex v whose degree is at least $x + 1$. Supplementing any k -path vertex cover of the graph $G' = G - v$ with v we obtain a k -path vertex cover of G . Here G' has order $n' = n - 1$ and size $m' \leq m - x - 1$. Hence, assuming by induction that (a', b') is a feasible pair for the smaller graph G' , we obtain:

$$\begin{aligned} \psi_k(G) &\leq 1 + \psi_k(G') \leq 1 + a'(n - 1) + b'(m - x - 1) \\ &= a'n + b'm - (a + qx + (b - w)(x + 1) - 1) \\ &\leq a'n + b'm. \end{aligned}$$

□

Based on this theorem, one can derive an infinite class of feasible pairs (a', b') already from one single pair (a, b) . Nevertheless, we think that the most interesting case occurs when the two upper-bound conditions on w coincide. This special value is the one exposed in Theorem 6 above. More explicitly, it is obtained by substituting $q = \frac{(a+b+bx-1)}{x+2}$ and $w = 2q$.

3 Estimates in terms of vertex degrees

In this section we give general estimations on $\psi_k(G)$ in terms of the order and of the minimum, maximum, and average vertex degrees.

3.1 In terms of δ and Δ

In the proof of the first general upper bound we will refer to the following decomposition theorem of Lovász.

Theorem 11 ([18]) *If a and b are nonnegative integers and G is a graph of maximum degree at most $a+b+1$, then the vertex set of G can be partitioned into two sets which induce subgraphs of maximum degree at most a and b .*

The following theorem was proved by Brešar *et al.* [5] in 2011.

Theorem 12 ([5]) *Let G be a graph of order n and of maximum degree Δ . Then*

$$\psi_3(G) \leq \frac{\lceil \frac{\Delta-1}{2} \rceil}{\lceil \frac{\Delta+1}{2} \rceil} n.$$

By definition, $\psi_k(G) \leq \psi_3(G)$ if $k \geq 3$. Theorem 12 therefore implies $\psi_k(G) \leq \frac{\Delta-1}{\Delta+1} n$ for every $k \geq 3$ if Δ is odd and $\psi_k(G) \leq \frac{\Delta}{\Delta+2} n$ if Δ is even. For $k = 3$ and any odd Δ the bound is tight, as shown by the complete graphs $K_{\Delta+1}$. Similarly, omitting a perfect matching from $K_{\Delta+2}$ we obtain tight examples for all even Δ with $k = 3$.

Our next result shows that the upper bound can be improved for all $k \geq 4$ and all $\Delta \geq 4$. Note that part (i) of Theorem 13 gives the same bound as Theorem 12 for $k = 3$ and that the bound stated for $\Delta = 2$ here is exactly the same as the one in Proposition 8.

Theorem 13 *Let k and Δ be integers with $k \geq 3$ and $\Delta = 2$ or $\Delta \geq 4$, and let G be a graph of order n and of maximum degree at most Δ . Then the following hold.*

(i) *If $\Delta \geq 2$ is even, then*

$$\psi_k(G) \leq \frac{(k-1)(\Delta-2)+4}{(k-1)\Delta+4} n.$$

(ii) If $\Delta \geq 5$ is odd, then

$$\psi_k(G) \leq \frac{(k-1)(\Delta-3)+8}{(k-1)(\Delta-1)+8} n.$$

Proof. We fix an integer $k \geq 3$ and proceed by induction on Δ . If $\Delta(G) \leq 2$ then, by Proposition 8, $\psi_k(G) \leq \frac{2}{k+1} n$ holds. This verifies the basic case of (i). Now, assume that Δ is even, $\Delta \geq 4$, and that G has maximum degree at most Δ . By Theorem 11, $V(G)$ can be partitioned into two sets A and B , such that $\Delta(G[A]) \leq \Delta - 2$ and $\Delta(G[B]) \leq 1$. Since $G[B]$ is P_k -free, $\psi_k(G) \leq |A|$ holds. On the other hand, if T is a minimum k -path vertex cover in $G[A]$, then $B \cup T$ covers all paths of order k in G , and we have $\psi_k(G) \leq |B| + |T|$. If $|A| \leq \frac{(k-1)(\Delta-2)+4}{(k-1)\Delta+4} n$ holds, then $\psi_k(G) \leq |A|$ establishes the upper bound. In the other case, we observe that

$$\begin{aligned} \psi_k(G) &\leq |B| + |T| \leq n - |A| + \frac{(k-1)(\Delta-4)+4}{(k-1)(\Delta-2)+4} |A| \\ &= n - \frac{2(k-1)}{(k-1)(\Delta-2)+4} |A| < \frac{(k-1)(\Delta-2)+4}{(k-1)\Delta+4} n. \end{aligned}$$

This proves (i).

Suppose next that Δ is odd and $\Delta \geq 5$. Say, $\Delta = 2a + 2b + 1$ where a, b are positive integers. From Theorem 11 we know that there exists a vertex partition $V(G) = A \cup B$ such that $\Delta(G[A]) \leq 2a$ and $\Delta(G[B]) \leq 2b$. Moreover, by (i), there exist k -path vertex covers T_A in $G[A]$ and T_B in $G[B]$ with

$$|T_A| \leq \frac{(k-1)(a-1)+2}{(k-1)a+2} |A|, \quad |T_B| \leq \frac{(k-1)(b-1)+2}{(k-1)b+2} |B|.$$

Both $T_A \cup B$ and $T_B \cup A$ are k -path transversals of G . Hence, denoting $x = |A|/n$ and $y = |B|/n = 1 - x$, we have

$$\frac{\psi_k(G)}{n} \leq \min \left(\frac{(k-1)(a-1)+2}{(k-1)a+2} x + (1-x), \frac{(k-1)(b-1)+2}{(k-1)b+2} (1-x) + x \right).$$

The first term of minimization is a decreasing function of x while the second term is an increasing function. Hence, the upper bound never exceeds the one obtained when the two numbers are equal, that is

$$\begin{aligned} \frac{(k-1)(a-1)+2}{(k-1)a+2} x + (1-x) &= \frac{(k-1)(b-1)+2}{(k-1)b+2} (1-x) + x, & (3) \\ \frac{k-1}{(k-1)b+2} &= \left(\frac{k-1}{(k-1)a+2} + \frac{k-1}{(k-1)b+2} \right) x, \\ 1 &= \left(\frac{(k-1)b+2}{(k-1)a+2} + 1 \right) x. \end{aligned}$$

Recalling that $2a + 2b = \Delta - 1$, the worst case occurs when

$$x = \frac{(k-1)(2a) + 4}{(k-1)(\Delta-1) + 8}, \quad y = \frac{(k-1)(2b) + 4}{(k-1)(\Delta-1) + 8}.$$

Thus, substituting this particular value of x into the left-hand side of (3) we obtain

$$\begin{aligned} \frac{\psi_k(G)}{n} &\leq \frac{[(k-1)(2a-2) + 4] + [(k-1)(2b) + 4]}{(k-1)(\Delta-1) + 8} \\ &= \frac{(k-1)(\Delta-3) + 8}{(k-1)(\Delta-1) + 8}. \end{aligned}$$

This proves (ii) and completes the proof of the theorem. \square

For $\Delta = 2, 4, 5, 6, 7, 8, 9, \dots$ the following sequence is obtained for the coefficients of n , which is strictly increasing for every $k \geq 4$:

$$\frac{1}{2} < \frac{k+1}{2k} < \frac{k+3}{2k+2} < \frac{2k}{3k-1} < \frac{2k+2}{3k+1} < \frac{3k-1}{4k-2} < \frac{3k+1}{4k} < \dots$$

To derive a general lower bound in terms of minimum and maximum degree, we need the following extremal result due to Erdős and Gallai.

Theorem 14 ([8]) *If G is a graph on n vertices that does not contain a path of order k , then it cannot have more than $\frac{n(k-2)}{2}$ edges. Moreover, the bound is achieved when the graph consists of vertex-disjoint cliques on $k-1$ vertices.*

Now we are in a position to extend the inequality (1) from regular graphs to arbitrary graphs, using the approach in [4].

Theorem 15 *Let G be a graph of order n with minimum degree δ and maximum degree Δ . If $k \geq 3$ and $\delta \geq k-1$, then*

$$\psi_k(G) \geq \frac{\delta - k + 2}{\delta + \Delta - k + 2} n.$$

Proof. Let $T \subseteq V(G)$ be a minimum k -path vertex cover of G , i.e. $|T| = \psi_k(G)$, and let $\bar{T} = V(G) - T$. Further, let E_T and $E_{\bar{T}}$ be the set of edges induced by T and by \bar{T} , respectively, and $E_{T\bar{T}}$ be the set of edges with one endvertex in T and the other in \bar{T} . We observe that

$$\sum_{v \in V(G)} d(v) = 2|E(G)| = 2|E_T| + 2|E_{T\bar{T}}| + 2|E_{\bar{T}}|$$

and also that $\Delta \cdot |T| \geq 2|E_T| + |E_{T\bar{T}}|$ and $\delta \cdot |\bar{T}| \leq |E_{T\bar{T}}| + 2|E_{\bar{T}}|$. Since the subgraph induced by \bar{T} with edge set $E_{\bar{T}}$ does not contain a path P_k , Theorem 14 implies $|E_{\bar{T}}| \leq \frac{|\bar{T}|(k-2)}{2}$. From these formulas we get

$$\begin{aligned} |T| &\geq \frac{1}{\Delta} (2|E_T| + |E_{T\bar{T}}|) \geq \frac{1}{\Delta} |E_{T\bar{T}}| \geq \frac{1}{\Delta} (\delta|\bar{T}| - 2|E_{\bar{T}}|) \\ &\geq \frac{1}{\Delta} (\delta|\bar{T}| - |\bar{T}|(k-2)) = \frac{\delta - k + 2}{\Delta} |\bar{T}|. \end{aligned}$$

It follows that

$$n = |T| + |\bar{T}| \leq |T| + \frac{\Delta}{\delta - k + 2} |T| = \frac{\delta + \Delta - k + 2}{\delta - k + 2} |T|$$

and

$$\psi_k(G) = |T| \geq \frac{\delta - k + 2}{\delta + \Delta - k + 2} n.$$

□

3.2 In terms of degree sequence and average degree

In this subsection, we improve the following upper bound that was proved in [5]:

$$\psi_k(G) \leq n - \frac{2k-2}{k} \sum_{v \in V(G)} \frac{1}{1+d(v)}.$$

Theorem 16 *Let G be a graph of order n , without isolated vertices. Then for every integer $k \geq 3$ we have*

$$\psi_k(G) \leq n - \frac{2k-3}{k-1} \sum_{v \in V(G)} \frac{1}{1+d(v)}.$$

Proof. We take an ordering v_1, v_2, \dots, v_n of the vertex set at random, each of the $n!$ orders being equally likely. Having an ordering, we define a weight function $w : V(G) \rightarrow \{0, \frac{k-2}{k-1}, 1\}$ as follows. For each vertex v_i , let $w(v_i) = 1$ if v_i has no neighbor among v_1, \dots, v_{i-1} ; let $w(v_i) = (k-2)/(k-1)$ if v_i has exactly one neighbor with an index smaller than i ; and let $w(v_i) = 0$ otherwise. Define Y as the set of vertices assigned with positive weights. Hence, a vertex v belongs to Y if and only if it appears as first or second vertex when the order is restricted to the closed neighborhood $N[v]$ of v . Since every order has the same probability, also the permutation of $N[v]$ is a random one and thus, each of the $d(v) + 1$ positions is equally likely to be taken by v . Consequently the expected total weight is

$$E = \left(1 + \frac{k-2}{k-1}\right) \sum_{v \in V(G)} \frac{1}{1+d(v)}.$$

Expectation of a random variable means a certain average over all outcomes of the event, thus there must exist a vertex order under which the total weight on the vertices of Y is at least E . Observe that $G[Y]$ is a forest. Indeed, should a cycle occur, its vertex of largest index would have at least two neighbors preceding it, a contradiction to the definition of Y . Remark further that any component of $G[Y]$ may contain at most one vertex of weight 1. Otherwise adding an edge between two such vertices would result in the same set Y and the subgraph induced by Y would contain a cycle which is a contradiction.

Now, we prove that $G[Y]$ contains a set S of vertices such that $G[S]$ is P_k -free and $|S| \geq \sum_{v \in Y} w(v)$. It suffices to prove the analogous statement for the components of $G[Y]$. If $G[Y_i]$ is a component that does not contain any P_k -subgraphs, we simply set $S_i = Y_i$ and $|S_i| \geq \sum_{v \in Y_i} w(v)$ clearly holds. Otherwise, consider a minimum k -path vertex cover T_i and its complement S_i in the tree $G[Y_i]$. By Theorem 2, we have $|S_i| \geq (k-1)|T_i|$ and under the present assumption $|S_i| \geq k-1$ holds. Moreover, at most one vertex in Y_i may have a weight of 1. We infer the following:

$$\begin{aligned} \sum_{v \in Y_i} w(v) &= \sum_{v \in T_i} w(v) + \sum_{v \in S_i} w(v) \leq \frac{k-2}{k-1} |T_i| + \frac{k-2}{k-1} |S_i| + \frac{1}{k-1} \\ &\leq \left(\frac{k-2}{(k-1)^2} + \frac{k-2}{k-1} \right) |S_i| + \frac{1}{k-1} = |S_i| - \frac{1}{(k-1)^2} |S_i| + \frac{1}{k-1} \leq |S_i|. \end{aligned}$$

The set $S = \bigcup S_i$ induces a P_k -free subgraph in G and satisfies

$$|S| \geq \sum_{v \in V(G)} w(v) \geq E \geq \frac{2k-3}{k-1} \sum_{v \in V(G)} \frac{1}{1+d(v)}.$$

Since the complement $T = V(G) \setminus S$ is a k -path vertex cover in G , we conclude that

$$\psi_k(G) \leq |T| \leq n - \frac{2k-3}{k-1} \sum_{v \in V(G)} \frac{1}{1+d(v)}.$$

□

Corollary 17 *If G is an isolate-free graph of order n and average degree \bar{d} , then*

$$\psi_k(G) \leq \left(1 - \frac{1}{\bar{d}+1} \cdot \frac{2k-3}{k-1} \right) n$$

for every $k \geq 3$.

4 Chordal graphs

By definition, a graph is chordal if it does not contain any induced cycles on more than three vertices. In this section we consider the well-studied class of chordal graphs and prove upper bounds on ψ_k for its members, in terms of the chromatic number and the order.

Theorem 18 *If $k \geq 2$ and G is a chordal graph of order n and chromatic number χ , then*

$$\psi_k(G) \leq \left(1 - \frac{2}{\chi} \cdot \frac{k-1}{k}\right) n.$$

Moreover, this bound is tight for every k with $\chi = 2$ and also for every χ with $k = 2$.

Proof. Consider a χ -coloring of G , and let X_1 and X_2 be the two largest color classes. Clearly, $|X_1| + |X_2| \geq \frac{2}{\chi} n$. The subgraph induced by $X_1 \cup X_2$ is bipartite and chordal. So, $G[X_1 \cup X_2]$ is a forest, and it has a k -path vertex cover T with $|T| \leq (|X_1| + |X_2|)/k$. Since $(V(G) \setminus (X_1 \cup X_2)) \cup T$ is a k -path vertex cover in G , we have

$$\begin{aligned} \psi_k(G) &\leq n - (|X_1| + |X_2|) + \frac{1}{k} (|X_1| + |X_2|) \\ &\leq n - \frac{k-1}{k} (|X_1| + |X_2|) \\ &\leq \left(1 - \frac{2}{\chi} \cdot \frac{k-1}{k}\right) n. \end{aligned}$$

Let now n be a multiple of $k\chi$. If $G = P_n$ (hence $\chi = 2$), then the equality $\psi_k(G) = n/k$ holds for any k because a vertex has to be chosen into a k -path vertex cover from any k consecutive vertices of P_n . Also, for $k = 2$ with arbitrary χ let $G = P_n^{\chi-1}$, the graph obtained from P_n by joining any two vertices whose distance along the path is less than χ . Then $\psi_2(G) = n - n/\chi$ because $\chi - 1$ vertices have to be chosen into a 2-path vertex cover from any χ consecutive vertices of $P_n^{\chi-1}$. \square

The above inequality is not tight in general; an improvement for almost all pairs k, χ (but not for all, cf. Table 1) is given by the following result. We state it in terms of the clique number, due to the nature of its proof. But every chordal graph G satisfies $\chi(G) = \omega(G)$, so the two theorems might be formulated in the same way.

Theorem 19 *If $k \geq 3$ and G is a chordal graph on n vertices with clique number ω , then*

$$\psi_k(G) \leq \frac{\omega}{\omega + k - 1} n.$$

Proof. The validity of the assertion is easy to see if $n \leq \omega + k - 1$ because deleting $n - k + 1$ vertices no room remains for P_k , hence $\psi_k(G) \leq n - k + 1 \leq \frac{\omega}{\omega + k - 1} n$ if n is so small. In the rest of the proof we assume $n > \omega + k - 1$ and apply induction on n .

The following argument is inspired by the theory of treewidth, and in particular by the notion of nice tree decomposition, but the formalism here is simpler because the graph G is assumed to be chordal. A detailed general treatment of tree decompositions can be found e.g. in the monograph [13], therefore we shall adopt facts from it without further references.

Let v_1, \dots, v_n be the vertices of G . Then there exists a rooted binary tree, say F , and subtrees F_1, \dots, F_n in it, where subtree F_i will represent vertex v_i for $i = 1, \dots, n$, with the following properties. For every node x of F , let us denote $S_x = \{i \mid x \in F_i\}$.

$\chi = \omega$	2	3	4	5	2	3	4	5
$k = 2$	1/2	2/3	3/4	4/5	2/3	3/4	4/5	5/6
$k = 3$	1/3	5/9	2/3	11/15	1/2	3/5	2/3	5/7
$k = 4$	1/4	1/2	5/8	7/10	2/5	1/2	4/7	5/8
$k = 5$	1/5	7/15	3/5	17/25	1/3	3/7	1/2	5/9
$k = 6$	1/6	4/9	7/12	2/3	2/7	3/8	4/9	1/2

Table 1: Comparison of Theorem 18 and Theorem 19, indicating best current bounds

- $V(F_i) \cap V(F_j) \neq \emptyset$ if and only if $v_i v_j \in E(G)$.
- $|S_x| \leq \omega$ for all $x \in V(F)$.
- If x has two children x' and x'' , then $S_x = S_{x'} = S_{x''}$.
- If x has one child x' , then $S_x \subset S_{x'}$ or $S_{x'} \subset S_x$, moreover $|S_x| = |S_{x'}| + 1$ or $|S_x| = |S_{x'}| - 1$.

For every $x \in V(F)$ let V_x denote the set of those vertices v_i for which there exists an x' in the subtree rooted at x in F such that $i \in S_{x'}$. Then we have, in particular, that $|V_x| = n > \omega + k - 1$ if x is the root of F , while $|V_x| = |S_x| \leq \omega$ if x is a leaf. By the property expressed in the fourth bullet above, $|V_x|$ can increase by at most 1 when we move from a node to its parent towards the root. Thus we can choose a node x such that $|V_x| = \omega + k - 1$. Now we put all $v \in S_x$ into the k -path vertex cover T to be constructed, delete the entire V_x from G , moreover omit those F_i which belong to vertices $v_i \in V_x$. After this, we keep only those nodes of F which are incident with at least one of the remaining subtrees.

Observe that the F_j not containing x either are entirely in the subtree rooted at x (hence they are omitted during this step) or are disjoint from the subtree rooted at x . Consequently the vertex set corresponding to S_x is a vertex cut in G , and there is no edge from $V_x \setminus T$ to $G - V_x$. It follows that $G[V_x \setminus T]$ is P_k -free, and every k -path vertex cover of $G - V_x$ is completed to a k -path vertex cover of G with the current vertices of T . Thus, applying the induction hypothesis we obtain

$$\psi_k(G) \leq |S_x| + \frac{\omega}{\omega + k - 1} (n - |V_x|) \leq \frac{\omega}{\omega + k - 1} n.$$

□

5 Concluding remarks and open problems

In this concluding section we discuss a method which leads to further improved upper bounds, study the case of planar graphs, and raise some problems which remain open for future research.

5.1 Bounds from maximum degree

An interesting aspect of Theorem 13 is that it offers a self-improving scheme. We illustrate this with the case of $\Delta = 11$.

Theorem 20 *If G is a graph on n vertices and $\Delta(G) = 11$, then*

$$\psi_k(G) \leq \frac{3k+5}{4k+4} n$$

for every $k \geq 6$.

Proof. By Theorem 11, G admits a vertex partition $A \cup B$ such that both $G[A]$ and $G[B]$ have maximum degree at most 5. Assume without loss of generality that $|A| \leq n/2 \leq |B|$ holds, and let T_B be a k -path vertex cover in $G[B]$. By Theorem 13 we can ensure

$$|T_B| \leq \frac{k+3}{2k+2} |B| = |B| - \frac{k-1}{2k+2} |B| \leq |B| - \frac{k-1}{4k+4} n.$$

Since $A \cup T_B$ is a k -path vertex cover, and $|A| + |B| = n$, the claimed upper bound follows. \square

Certainly the inequality in the theorem is valid also for $k \leq 5$, but to beat the earlier formula $\frac{4k}{5k-1} n$ we need $(k-1)(k-5) > 0$.

It is not clear at the moment, how strong upper bound can be derived as a limit of this approach as Δ gets large, and how restricted it will be as regards the value of k .

We should also note that it was necessary to exclude $\Delta = 3$ from the range of validity in Theorem 13. Indeed, the formula in (ii) would then yield $\frac{4}{k+3} n$, i.e., the coefficient of n would tend to zero as k gets large. But this would not be a valid formula, as the next result shows.

Theorem 21 *For every k , there are infinitely many 3-regular connected graphs G such that $\psi_k(G) > \frac{1}{4} |V(G)|$.*

Proof. Fix any $k \geq 3$. It follows from results of Erdős and Sachs [9] that for every even $n \geq 2^k$ there exists a 3-regular graph G of order n such that the length of the shortest cycle (i.e., the girth of G) is larger than k . Let now T be any k -path vertex cover in G , and consider the induced subgraph $H = G - T$. Say, H has connected components H_1, \dots, H_s . Each H_i is a tree (of diameter at most $k-2$). Regularity of degree 3 implies that there are exactly $|V(H_i)| + 2$ edges — that is, more than the order of H_i — from $V(H_i)$ to $V(G) \setminus V(H_i)$. Altogether more than $|V(H)|$ edges go from H to the vertices of T , while at most $3|T|$ edges go from T to H . It follows that $|T| > \frac{1}{3} |V(H)|$ and hence $|T| > \frac{1}{4} n$. \square

5.2 Planar graphs

In this subsection we present some partial results related to a conjecture which was first mentioned by Brešar et al. in 2011. Note that it is easy to construct planar graphs that satisfy the relation with equality, e.g. the vertex-disjoint union of octahedron graphs, adjacent in any planar way.

Conjecture 22 ([5]) *If G is a planar graph of order n , then $\psi_3(G) \leq \frac{2}{3}n$.*

It is a basic fact that $\omega(G) \leq 4$ holds for every planar graph. Then, Theorem 19 allows us to conclude that Conjecture 22 is true for those planar graphs which are also chordal. We may also derive upper bounds on $\psi_k(G)$ for $k \geq 4$.

Corollary 23 *If G is a graph of order n which is chordal and planar, then*

$$\psi_3(G) \leq \frac{2}{3}n, \quad \psi_4(G) \leq \frac{4}{7}n, \quad \text{and} \quad \psi_5(G) \leq \frac{1}{2}n.$$

On the class of K_3 -free planar graphs Theorem 5 and the inequality $m \leq 2n - 4$ together imply

$$\psi_3(G) \leq \frac{4n + m}{9} \leq \frac{4n + 2n - 4}{9} = \frac{2}{3}n - \frac{4}{9}.$$

Therefore, Conjecture 22 is also true for K_3 -free planar graphs. Moreover, Corollary 28 will show that the upper bound $2n/3$ can be considerably improved on this graph class.

The *forest number* $a(G)$ of a graph G is defined as the maximum number of vertices in an induced forest of G . As it is mentioned shortly in the concluding section of [5], there is an important relation between the k -path vertex cover number and the forest number. For the sake of completeness, we present a proof for this statement.

Proposition 24 ([5]) *For every integer $k \geq 3$ and for every graph G of order n ,*

$$\psi_k(G) \leq n - \frac{k-1}{k} a(G).$$

Proof. Consider a set S of vertices such that $|S| = a(G)$ and $G[S]$ is a forest. By Proposition 2, every k -path inside $G[S]$ can be covered by a set T of at most $|S|/k$ vertices. The set $(V(G) \setminus S) \cup T$ is clearly a k -path vertex cover in G . Hence, we have

$$\psi_k(G) \leq n - |S| + \frac{|S|}{k} = n - \frac{k-1}{k}|S| = n - \frac{k-1}{k} a(G).$$

□

The following is a famous unsolved problem on the forest number of planar graphs.

Albertson–Berman Conjecture ([1]) *If G is a planar graph of order n , then $a(G) \geq n/2$.*

Proposition 24 shows that the Albertson–Berman Conjecture, if true, implies Conjecture 22. Combining Proposition 24 and the following two results we may also conclude some upper bounds on $\psi_k(G)$.

Theorem 25 ([2]) *If G is a planar graph of order n , then $a(G) \geq \frac{2}{5}n$.*

Theorem 26 ([14]) *If G is a planar graph of order n that is K_3 -free, then $a(G) \geq \frac{71}{128}n + \frac{9}{16}$.*

Corollary 27 *If G is a planar graph of order n , then*

$$\psi_3(G) \leq \frac{11}{15}n < 0.734n$$

and

$$\psi_6(G) \leq \frac{2}{3}n.$$

Corollary 28 *If G is a planar and K_3 -free graph on n vertices, then*

$$\psi_3(G) \leq \frac{121}{192}n - \frac{3}{8}.$$

5.3 Further conjectures and open problems

Some of the estimates proved in this paper are tight, while some others probably aren't. In particular, on the class of chordal graphs we expect that the following improvement can be made.

Conjecture 29 *If $k \geq 3$ and G is a chordal graph with clique number ω and $n = |V(G)|$, then*

$$\psi_k(G) \leq \frac{\omega - 1}{\omega + k - 2}n.$$

The analogous inequality for $k = 2$ can easily be seen for every graph in which the equality $\chi = \omega$ is valid — hence for all perfect graphs and in particular for every chordal graph — by taking a proper vertex coloring with ω colors and deleting the largest color class.

The following further problems arise in a natural way in connection with our results on planar graphs.

Problem 30 *Determine the smallest constants $c_{1,k}$ and $c_{2,k}$ such that*

- (i) *if G is planar, then $\psi_k(G) \leq (c_{1,k} + o(1))n$;*
- (ii) *if G is planar and K_3 -free, then $\psi_k(G) \leq (c_{2,k} + o(1))n$,*

as $n \rightarrow \infty$.

Note that Conjecture 22 formulates the explicit guess $c_{1,3} = 2/3$.

Corollary 7 offers various upper bounds on $\psi_3(G)$. The following problem naturally arises.

Problem 31 *Are the upper bounds stated in Corollary 7 sharp?*

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