Relating the total domination number and the annihilation number of cactus graphs and block graphs

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Abstract

The total domination number $\gamma_t(G)$ of a graph G is the order of a smallest set $D \subseteq V(G)$ such that each vertex of G is adjacent to some vertex in D. The annihilation number $a(G)$ of G is the largest integer k such that there exist k different vertices in G with degree sum of at most $|E(G)|$. It is conjectured that $\gamma_t(G) \leq a(G) + 1$ holds for every nontrivial connected graph G. Here we establish the conjecture for cactus graphs and block graphs.

Key words: Total domination number, Annihilation number, Cactus graph, Block graph.

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1 Introduction

All graphs considered in this paper are nontrivial, finite, simple and undirected. By a nontrivial graph we mean a graph on at least two vertices. If $G = (V, E)$ is a graph, then $V = V(G)$ is the set of vertices of order $n(G) = |V|$, and $E = E(G)$ is the set of edges of size $m(G) = |E|$. The *degree* of a vertex $v \in V$ in graph G will be denoted by $d_G(v)$. A vertex v of degree 1 is a *leaf*, while its only neighbor is called a *support vertex*. If u has at least two neighbors which are leaves, then u is referred to as a *strong support vertex*. The minimum and maximum degree among the vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a $v \in V(G)$, the set of its neighbors is denoted by $N_G(v)$ and called the *open neighborhood of* v. We use a similar notation for a set $A \subseteq V(G)$, it is defined as $N_G(A) = \bigcup_{v\in A} N_G(v)$. If G is clear from the context, we simply write $d(v)$, $N(v)$ and $N(A)$ instead of $d_G(v)$, $N_G(v)$ and $N_G(A)$, respectively.

For a graph G a set $D \subseteq V(G)$ is a *total dominating set* if every $v \in V(G)$ has at least one neighbor in D; i.e., if $N(D) = V(G)$. If G does not contain isolated vertices, such a set

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D always exists, and the minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is the *total domination number* of G. A survey on the total domination can be found in [8], and more recently, the topic was thoroughly covered in the book [9]. It is easy to check that following formulas determine the total domination number of a cycle C_n of length $n \geq 3$:

$$\gamma_t(C_n) = \begin{cases} \frac{n}{2}+1, & \text{if} \quad n \equiv 2 \pmod 4; \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

For a set $B \subseteq V$ we define the graph $G - B$ as the graph obtained from G by deleting the vertices in B and all edges incident with them. Moreover, if $u_1u_2 \in E$ and $v_1v_2 \notin E$, we use the notations $G - u_1u_2$ and $G + v_1v_2$ for the graphs $(V, E \setminus \{u_1u_2\})$ and $(V, E \cup \{v_1v_2\})$, respectively. Let G_1 and G_2 be two vertex-disjoint graphs and let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. The *identification of vertices* v_1 and v_2 results in a graph G with $V(G) = (V(G_1) \cup V(G_2) \cup \{v\}) \setminus \{v_1, v_2\}$ such that $N_G(v) = N_{G_1}(v_1) \cup N_{G_2}(v_2)$. Moreover, for any vertex $u \neq v$, the open neighborhood remains the same.

The *subdivided star* $S(K_{1,\ell})$ is the graph on $2\ell + 1$ vertices which is constructed from the star $K_{1,\ell}$ by subdividing each edge exactly once (left-hand side of Figure 1). The *paw* is the graph P obtained from K_4 by deleting two neighboring edges (right-hand side of Figure 1). A connected graph is called *cactus graph* if its cycles are pairwise edge-disjoint. Moreover, G is a *block graph* if each 2-connected component of G is a clique.

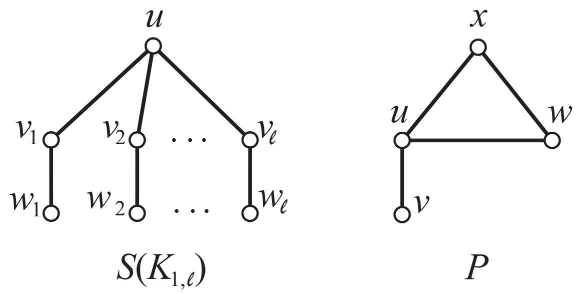

Figure 1: The subdivided star $S(K_{1,\ell})$, $\ell \geq 2$, and the paw graph P

For a subset $S \subseteq V(G)$ we define

$$\sum(S, G) = \sum_{v \in S} d_G(v).$$

Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G such that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$. The *annihilation number* $a(G)$ is the largest integer k such that $\sum_{i=1}^{k} d(v_i) \leq m(G)$. Equivalently, $a = a(G)$ is the only integer satisfying both

$$\sum_{i=1}^{a} d(v_i) \leq m(G) \quad \text{and} \quad \sum_{i=1}^{a+1} d(v_i) \geq m(G) + 1.$$

It is clear by definition that every independent set[1] satisfies $\sum_{v \in A} d(v) \leq m(G)$ and consequently, the annihilation number is an upper bound on the independence number [12]. The

[1] A set $A \subseteq V(G)$ is called an independent set if it induces an edgeless subgraph in G. The largest cardinality of such a vertex set is the independence number of G and denoted by $\alpha(G)$.

annihilation number was first introduced by Pepper in [11]. The 'annihilation process', which is referred to in this original definition, is very similar to the 'Havel-Hakimi process' (see [7] and [12] for exact descriptions).

In general, a set S of vertices is called an *annihilation set* if $\sum_{v\in S} d(v) \leq m(G)$; and S is an *optimal annihilation set*, if

$$|S| = a(G) \quad \text{and} \quad \max\{d(v) \mid v \in S\} \leq \min\{d(u) \mid u \in V(G) \setminus S\}.$$

In particular, if G is a connected graph on at least 3 vertices, any optimal annihilation set of G contains all leaves.

Assuming that S is an optimal annihilation set, we introduce the following notations. First, denote by $d^*(G)$ (or simply by d^*) the minimum vertex degree over the set $V(G) \setminus S$. Note that $d^*(G) = d(v_{a(G)+1})$, and consequently, the value of $d^*(G)$ is independent from the choice of the optimal annihilation set S.

The following conjecture can be found in a slightly different form in Graffiti.pc [6], and was later reformulated in [4].

Conjecture 1 [4, 6] *If G is a connected nontrivial graph, then*

$$\gamma_t(G) \leq a(G) + 1. \tag{1}$$

By definition, every graph satisfies $a(G) \geq \left\lfloor \frac{n(G)}{2} \right\rfloor$. Hence, the formulas given for $\gamma_t(C_n)$ above show that each cycle C_n satisfies the conjecture. Further, if $\delta(G) \geq 3$, it was observed that the total domination number is at most $\left\lfloor \frac{n(G)}{2} \right\rfloor$ [1, 3, 13, 14]. Hence, if $\delta(G) \geq 3$, then $\gamma_t(G) \leq a(G)$ clearly holds, even if G is disconnected. Therefore, it is interesting to study this conjecture for graphs with small minimum degree, i.e. $\delta(G) \in \{1, 2\}$. So far, Conjecture 1 has been proved for only one further important graph class. The following result was established by Desormeaux, Haynes, and Henning in 2013.

Theorem 2 [4] *If T is a nontrivial tree, then $\gamma_t(T) \leq a(T) + 1$, and the bound is sharp.*

A similar result was proved by Desormeaux, Henning, Rall, and Yeo [5] for the 2-domination number of trees. Very recently, a different proof was given for the same statement by Lyle and Patterson [10]. Namely, their result can be obtained if we replace the total domination number with the 2-domination number in Theorem 2.

In this paper we prove Conjecture 1 over two graph classes, namely for cactus graphs and block graphs. These are two natural generalizations of trees and also, for cactus graph we have $\delta \leq 2$ and there exist block graphs with small minimum degree. Remark that cactus and block graphs are well-studied classes with several applications, see for instance [2]. Our main results are the following ones.

Theorem 3 *If G is a nontrivial cactus graph, then $\gamma_t(G) \leq a(G) + 1$.*

Theorem 4 *If G is a nontrivial block graph, then $\gamma_t(G) \leq a(G) + 1$.*

To formulate and to prove our results we will use the following *function* f defined for every finite graph G as

$$f(G) = n(G) + 3m(G) + n_1(G)$$

where $n_1(G)$ denotes the number of leaves in G. Remark that f is strictly monotone in the sense that if G' is a proper subgraph of G, then $f(G') < f(G)$. Indeed, $n(G') + m(G') < n(G)+m(G)$ clearly holds, and $2m(G')+n_1(G') \le 2m(G)+n_1(G)$ is true because the deletion of an edge may result in at most two new leaves. Also note that we have $f(G) \ge 7$ for any nontrivial, finite and connected graph G.

The paper is organized as follows. In Section 2, we establish several lemmas which will be referred to in later proofs. In Section 3 and 4 we prove Theorem 3 and 4, respectively. In the last section we discuss the sharpness of our main theorems and arise some related problems.

2 Preliminary results

Here we present some preliminary results on how we can obtain a smaller graph G' from G (mainly, by deleting some edges and/or vertices from G) such that $\gamma_t(G') \le a(G')+1$ implies $\gamma_t(G) \le a(G) + 1$. First we consider changes related to vertices of small degree.

Lemma 5 *Assume that G is a connected graph on at least three vertices and it fulfills at least one of the following properties:*

- (i) *$d^*(G) \le 2$;*
- (ii) *G has a strong support vertex;*
- (iii) *G contains an induced path $vu_1u_2u_3w$ such that $d(u_1) = d(u_2) = d(u_3) = 2$;*
- (iv) *G contains a path $u_1u_2u_3v$ such that u_1 is a leaf and $d(u_2) = d(u_3) = 2$;*
- (v) *G contains two adjacent support vertices.*

Then, there exists a nontrivial connected graph G' with $f(G') < f(G)$ such that $\gamma_t(G') \le a(G')+1$ implies $\gamma_t(G) \le a(G)+1$. Moreover, if G is a cactus graph, then G' can be chosen to be a cactus graph as well; and if G is a block graph, in cases $(ii)-(v)$, G' can be chosen to be a block graph.

Proof. Since trees and cycles satisfy Conjecture 1, we may suppose throughout that G is neither a tree nor a cycle.

(i) First assume that $d^*(G) \le 2$. Since G is neither a tree nor a cycle, there exists a vertex $v \in V(G)$ with $d(v) \ge 3$ which is incident to a cycle. Let $e = vu$ be an edge from that cycle. Clearly, $G' = G - e$ is connected, $f(G') < f(G)$ and $m(G') = m(G) - 1$. The deletion of an edge does not decrease the total domination number. This establishes $\gamma_t(G) \le \gamma_t(G')$. Consider now an optimal annihilation set S' of G'. By definition, it satisfies $\sum(S', G') \le m(G') = m(G)-1$. If $u, v \notin S'$ then $\sum(S', G) = \sum(S', G') \le m(G)-1$; if S' contains exactly one of u and v, then $\sum(S', G) = \sum(S', G') + 1 \le m(G)$. In either case $a(G) \ge |S'| = a(G')$ follows. In the third case $u, v \in S'$ and $\sum(S', G) = \sum(S', G') + 2 \le m(G) + 1$. Let $V_{1,2}$ denote the set of vertices which have degree 1 or 2 in G. Our assumption $d^*(G) \le 2$ implies

$\sum(V_{1,2}, G) \geq m(G) + 1$. Since $d(v) \geq 3$, we have $\sum(V_{1,2} \cup \{v\}, G) \geq m(G) + 4$. Therefore, $(V_{1,2} \cup \{v\}) \nsubseteq S'$ implies that we have a vertex $v^* \in V_{1,2}$ which is not contained in S'. If v is replaced with v^* in S', we obtain a set S with $\sum(S, G) \leq \sum(S', G) - 1 \leq m(G)$. This proves $a(G) \geq |S| = a(G')$. If G' satisfies (1), we may conclude that the same is true for G:

$$\gamma_t(G) \leq \gamma_t(G') \leq a(G') + 1 \leq a(G) + 1.$$

In the continuation of the proof we will assume that $d^*(G) \geq 3$.

(ii) Assume that a vertex $v \in V(G)$ has two neighbors u_1 and u_2 which are leaves in G. Since v remains a support vertex in $G' = G - \{u_1\}$, it is contained in every total dominating set of G'. This implies $\gamma_t(G') = \gamma_t(G)$. On the other hand, every optimal annihilation set of G contains u_1 and hence $a(G') \leq a(G)$. Then, $f(G') < f(G)$, and $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$.

(iii) If $vu_1u_2u_3w$ is an induced path in G and $d(u_1) = d(u_2) = d(u_3) = 2$, consider the graph $G' = G - \{u_1, u_2, u_3\} + vw$. Observe that $n(G') = n(G) - 3$, $m(G') = m(G) - 3$, $n_1(G') = n_1(G)$ and hence, $f(G') = f(G) - 12$. Let D' be an optimal total dominating set of G' and define D as follows:

$$D = \begin{cases} D' \cup \{u_1, u_3\}, & \text{if} \quad v, w \in D'; \\ D' \cup \{u_2, u_3\}, & \text{if} \quad w \notin D'; \\ D' \cup \{u_1, u_2\}, & \text{if} \quad w \in D' \text{ and } v \notin D'. \end{cases}$$

In either case, D is a total dominating set in G. Hence, $\gamma_t(G) \leq \gamma_t(G') + 2$. Consider next an optimal annihilation set S' of G'. Since $d_G(v) = d_{G'}(v)$ and $d_G(w) = d_{G'}(w)$, $\sum(S', G) = \sum(S', G') \leq m(G') = m(G) - 3$. Our assumption $d^*(G) \geq 3$ implies that every vertex x with degree $d(x) \leq 2$ is contained in every optimal annihilation set of G. Hence, either $S' \cup \{u_1, u_2, u_3\}$ is a subset of an optimal annihilation set of G and $a(G) \geq a(G') + 3$, or there is a vertex $v^* \in S'$ with $d(v^*) \geq 3$. In the latter case, consider $S = (S' \setminus \{v^*\}) \cup \{u_1, u_2, u_3\}$, and observe that $\sum(S, G) \leq \sum(S', G) - 3 + 3 \cdot 2 \leq m(G)$. Therefore, $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. If G' satisfies inequality (1), these gives

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1$$

that proves the statement for property *(iii)*.

(iv) Let $u_1u_2u_3v$ be a path in G such that $d(u_1) = 1$ and $d(u_2) = d(u_3) = 2$. Since G is connected and not a path, $G' = G - \{u_1, u_2, u_3\}$ is nontrivial, and we have $f(G') < f(G)$. If D' is an optimal total dominating set of G', then $D = D' \cup \{u_2, u_3\}$ totally dominates all vertices in G. Thus, $\gamma_t(G) \leq |D| \leq \gamma_t(G') + 2$. Next, we choose an optimal annihilation set S' in G' and consider three cases concerning v and S'.

- If $d(v) = 2$, then G contains three consecutive degree-2 vertices and, as we have already proved it in *(iii)*, there exists a graph G' with the required property.

- If $v \notin S'$, then $\sum(S', G) = \sum(S', G')$, and $\sum(S', G) \leq m(G') = m(G) - 3$. Hence, $S = S' \cup \{u_1, u_2\}$ satisfies $\sum(S, G) = \sum(S', G) + 3 \leq m(G)$, and $a(G) \geq a(G') + 2$. These, together with the assumption $\gamma_t(G') \leq a(G') + 1$, establishes inequality (1) for G.

- In the last case we assume that both $v \in S'$ and $d(v) \geq 3$ hold. Then, $\sum(S', G) = \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 2$. We define $S = (S' \setminus \{v\}) \cup \{u_1, u_2, u_3\}$ and observe that $\sum(S, G) = \sum(S', G) - d(v) + 5 \leq m(G)$. Hence, S is an annihilation set in G and we may conclude $a(G) \geq |S| \geq a(G') + 2$. The statement of the lemma is proved by the following chain of inequalities: $\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1$.

(v) Let u and v be two leaves in G with support vertices u' and v' such that $uu', vv', u'v' \in E(G)$. Since G is not a path, at least one of these two support vertices, say u', is of degree of at least 3. Then, we define $G' = G - uu' + uv$ and observe that $f(G') = f(G) - 1$. Let D' be an optimal total dominating set of G'. Since v is a support vertex in G', $v \in D'$ must hold. Moreover, since $N_{G'}(u) \subseteq N_{G'}(v')$, we can choose D' such that u does not belong to it. Then, $D = (D' \setminus \{v\}) \cup \{u'\}$ is a total dominating set in G. Hence, $\gamma_t(G) \leq |D| = |D'| = \gamma_t(G')$. By construction, every vertex has the same degree in G as in G' with the two exceptions v and u', for which $d_G(u') = d_{G'}(u') + 1$ and $d_G(v) = d_{G'}(v) - 1$. Hence, any optimal annihilation set S' of G' satisfies one of the following cases.

- If $u', v \in S'$ or $u', v \notin S'$, then $\sum(S', G) = \sum(S', G') \leq m(G') = m(G)$. Therefore, $a(G) \geq |S'| = a(G')$.
- If $v \in S'$ and $u' \notin S'$, then $\sum(S', G) = \sum(S', G') - 1 \leq m(G') - 1 = m(G) - 1$. Therefore, $a(G) \geq |S'| = a(G')$.
- If $u' \in S'$ and $v \notin S'$, then $\sum(S', G) = \sum(S', G') + 1 \leq m(G') + 1 = m(G) + 1$. We define $S = (S' \setminus \{u'\}) \cup \{v\}$. By our assumption, $d_G(u') \geq 3$ and so, we have $\sum(S, G) = \sum(S', G) - d_G(u') + 1 \leq m(G) + 1 - d_G(u') + 1 \leq m(G) - 1$. This implies $a(G) \geq a(G')$.

We have seen that for all possible cases $a(G') \leq a(G)$ and $\gamma_t(G) \leq \gamma_t(G')$. Together with the condition that G' satisfies (1), these imply $\gamma_t(G) \leq \gamma_t(G') \leq a(G') + 1 \leq a(G) + 1$.

At the end of the proof we remark that all the above transformations result in a cactus graph G', if G was of the same type. Further, with the only exception of (i), the obtained graphs stay block graphs if G is a block graph. □

Lemma 6 *(i) For an integer $\ell \geq 3$, let $Q \cong K_\ell$ be a complete subgraph of the connected graph G such that Q contains exactly one vertex, say x, of degree larger than $\ell - 1$. Assume further that $G' = G - (V(Q) \setminus \{x\})$ satisfies $\gamma_t(G') \leq a(G') + 1$. Then, $\gamma_t(G) \leq a(G) + 1$ follows.*

(ii) Let C be a cycle in a connected graph G such that C contains exactly one vertex which is of degree larger than 2. Then, there exists a nontrivial connected graph G' with $f(G') < f(G)$ such that $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$. Moreover, if G is a cactus graph, then G' can be chosen to be a cactus graph as well.

Proof. (i) We suppose $d(x) \geq \ell \geq 3$ and $V(Q) = \{v_1, v_2, \ldots, v_{\ell-1}, x\}$. For any total dominating set D' of G', $D' \cup \{x\}$ is a total dominating set of G. Hence, $\gamma_t(G) \leq \gamma_t(G') + 1$. Now, let S' be an optimal annihilation set in G'.

- If $x \notin S'$, then $\sum(S',G) = \sum(S',G') \le m(G') = m(G) - \binom{\ell}{2}$. In this case, let $S = S' \cup \left\{v_1, \ldots, v_{\lfloor \frac{\ell}{2} \rfloor}\right\}$. Then

$$\sum(S,G) = \sum(S',G) + \left\lfloor \frac{\ell}{2} \right\rfloor (\ell - 1) \le m(G) - \binom{\ell}{2} + \left\lfloor \frac{\ell}{2} \right\rfloor (\ell - 1) \le m(G),$$

implying that $a(G) \ge |S| = a(G') + \lfloor \frac{\ell}{2} \rfloor$. Together with the conditions given in (i) for G' and ℓ, we have

$$\gamma_t(G) \le \gamma_t(G') + 1 \le a(G') + 2 \le a(G) + 2 - \left\lfloor \frac{\ell}{2} \right\rfloor \le a(G) + 1. \tag{2}$$

- If $x \in S'$, then $\sum(S',G) = \sum(S',G') + \ell - 1$. Hence, $\sum(S',G) \le m(G') + \ell - 1 = m(G) - \binom{\ell}{2} + \ell - 1$. Let $S = (S' \setminus \{x\}) \cup \left\{v_1, \ldots, v_{\lfloor \frac{\ell}{2} \rfloor + 1}\right\}$. Then,

$$\begin{aligned}\sum(S,G) &= \sum(S',G) - d_G(x) + \left(\left\lfloor \frac{\ell}{2} \right\rfloor + 1\right)(\ell - 1) \\ &\le m(G) - \binom{\ell}{2} + \ell - 1 - \ell + \left(\left\lfloor \frac{\ell}{2} \right\rfloor + 1\right)(\ell - 1) \\ &= m(G) + (\ell - 1)\left(\left\lfloor \frac{\ell}{2} \right\rfloor + 1 - \frac{\ell}{2}\right) - 1,\end{aligned}$$

and since $\ell - 1 \ge 2$ and $\lfloor \frac{\ell}{2} \rfloor + 1 - \frac{\ell}{2} \ge \frac{1}{2}$, we have $\sum(S,G) \le m(G)$. Therefore, $a(G) \ge |S| = a(G') + \lfloor \frac{\ell}{2} \rfloor$ and, again, the inequality chain (2) finishes the proof.

(ii) Since $C_3 = K_3$, it suffices to prove (ii) for cycles $C \cong C_\ell$ of length $\ell \ge 4$. If $d^*(G) \le 2$ or $\ell \ge 6$, Lemma 5(i) and (iii) establish the statement. Henceforth, we will suppose that $d^*(G) \ge 3$ and $\ell = 4$ or 5. Let $xv_1 \ldots v_{\ell-1}x$ be the cycle C such that $d(x) \ge 3$.

First, assume that $\ell + d_G(x) \ge 8$; i.e., at least one of $\ell = 5$ and $d_G(x) \ge 4$ holds. Let $G' = G - (V(C) \setminus \{x\})$ and let D' be an optimal total dominating set of G'. Observe that $D = D' \cup \{v_2, v_3\}$ is a total dominating set in G and consequently, $\gamma_t(G) \le \gamma_t(G') + 2$. Now, fix an optimal annihilation set S' in G' and consider the following two subcases.

- If $x \notin S'$, we have $\sum(S',G) = \sum(S',G') \le m(G') = m(G) - \ell$. Then, we define $S = S' \cup \{v_1, v_2\}$ and observe that $\sum(S,G) = \sum(S',G) + 2 \cdot 2 \le m(G) - \ell + 4 \le m(G)$. This proves $a(G) \ge |S| = a(G') + 2$.

- If $x \in S'$, we have $\sum(S',G) = \sum(S',G') + 2 \le m(G') = m(G) - \ell + 2$. In this case, consider $S = (S' \setminus \{x\}) \cup \{v_1, v_2, v_3\}$. For this set,

$$\sum(S,G) \le \sum(S',G) - d_G(x) + 3 \cdot 2 \le m(G) - \ell - d_G(x) + 8 \le m(G)$$

holds under the present assumption $\ell + d_G(x) \ge 8$. Therefore, we have $a(G) \ge |S| = a(G') + 2$.

In either subcase, if G' satisfies (1), we may conclude that

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

In the other case, $C \cong C_4$ and $d_G(x) = 3$. Here, we define $G' = G - V(C)$. Since $d_G(x) = 3$, G' is connected. If G' consists of only one vertex, $\gamma_t(G) = 2 < a(G) + 1$ can be proved directly. Hence, we may assume that G' is nontrivial. Let D' be an optimal total dominating set in G' and observe that, also in this case, $D = D' \cup \{v_2, v_3\}$ is a total dominating set in G. Hence, $\gamma_t(G) \leq \gamma_t(G') + 2$. On the other hand, let S' be an optimal annihilation set in G'. Since there is at most one edge between S' and $V(C)$, $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 4$. Moreover, for $S = S' \cup \{v_1, v_2\}$, we obtain $\sum(S, G) = \sum(S', G) + 4 \leq m(G)$, from which $a(G) \geq a(G') + 2$ follows. Thus, if G' satisfies (1), the desired inequality $\gamma_t(G) \leq a(G) + 1$ holds again. $\square$

The analogue of the following proof was given by Desormeaux et al. [4] inside the proof of Theorem 2. Here, we restate and prove it in a modified form.

Lemma 7 *Let H be a nontrivial connected graph and T be a tree such that $V(H) \cap V(T) = \emptyset$. Suppose that $w \in V(H)$, $u \in V(T)$, and v is a leaf in T such that $d(u, v) \geq 3$. If G is obtained from H and T by identifying w and u, there exists a connected graph G' with $f(G') < f(G)$ such that $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$.*

Proof. First note that the statement follows from Lemma 5(i) if $d^*(G) \leq 2$. Hence, we may suppose that $d^*(G) \geq 3$. Assume that T is rooted in u and choose a leaf $v_1 \in V(T)$ which is of maximum distance from u. Let v_2 be the parent of v_1, and v_3 be the parent of v_2. By assumption, $d(u, v_1) \geq 3$ and hence, $v_i \neq u$ $(i = 1, 2, 3)$.

We will consider graphs G' obtained from G by removing a set of vertices from $V(T)$ in such a way that G' will stay connected. Throughout, S' will denote an optimal annihilation set in G'.

If v_2 is a strong support vertex, Lemma 5(ii) implies the statement. So, we may suppose that v_1 is the only leaf of the support vertex v_2. Since v_1 is of maximum distance from u, $d(v_2) = 2$ also follows. Remark that the same is true for any other leaf and its support vertex, if the leaf is of maximum distance from u. Suppose that $d(v_3) \geq 3$ and let $G' = G - \{v_1, v_2\}$. So $m(G') = m(G) - 2$. If v_3 is a support vertex in G', then v_3 belongs to a minimum total dominating set D' of G'. If v_3 is not a support vertex, then every child of v_3 is a support vertex of degree 2. If a leaf-neighbor of a child of v_3 belongs to D', then we can simply replace it in D' with the vertex v_3. In either case, we may assume that $v_3 \in D'$. Thus the set $D = D' \cup \{v_2\}$ is a total dominating set of G, and so $\gamma_t(G) \leq |D| = |D'| + 1 = \gamma_t(G') + 1$. Independently of whether vertex v_3 lies in S' or not we have $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 1$. Consider $S = S' \cup \{v_1\}$. Then $\sum(S, G) = \sum(S', G) + d(v_1) \leq m(G)$, implying that $a(G) \geq |S| = |S'| + 1 = a(G') + 1$. By assumption, we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 1 \leq a(G') + 2 \leq a(G) + 1.$$

So, we may suppose that $d(v_3) = 2$. Now we have three consecutive vertices v_1, v_2, v_3 with degrees $d(v_1) = 1$ and $d(v_2) = d(v_3) = 2$. Thus, by Lemma 5(iv), there exists a graph G' with $f(G') < f(G)$ which satisfies the statement. $\square$

The following lemmas will be needed to cover two specific cases in the proofs of Theorems 3 and 4. Therefore, we give the proof for both cases here.

Lemma 8 *Let H and $F \cong S(K_{1,\ell})$ be two vertex-disjoint graphs with $n(H) \geq 3$ and $\ell \geq 2$. Assume that w is a vertex of H such that $H - \{w\}$ is connected and u is the central vertex of the subdivided star F. If G is the graph obtained from H and F by identifying w and u, and $\gamma_t(G - V(F)) \leq a(G - V(F)) + 1$, then $\gamma_t(G) \leq a(G) + 1$.*

Proof. Suppose the subgraph F of G is rooted in u. We denote with $v_1, \dots, v_\ell$ the children of u, and with $w_1, \dots, w_\ell$ the leaves. By our assumption, $G' = G - V(F) = G - \{u, v_1, \dots, v_\ell, w_1, \dots, w_\ell\}$ is a nontrivial connected graph, and $m(G') = m(G) - d_G(u) - \ell$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, v_1, \dots, v_\ell\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + \ell + 1 = \gamma_t(G') + \ell + 1$. Now, consider an optimal annihilation set S' in G'. Independently of whether the vertices in $N_{G'}(u)$ are inside S' or not, we have $\sum(S', G) \leq \sum(S', G') + d_G(u) - \ell \leq m(G') + d_G(u) - \ell = m(G) - 2\ell$. Let $S = S' \cup \{v_1, w_1, \dots, w_\ell\}$. Then $\sum(S, G) = \sum(S', G) + 2 + \ell \leq m(G) - 2\ell + \ell + 2 \leq m(G)$, since $\ell \geq 2$. Then, we have $a(G) \geq |S| = |S'| + \ell + 1 = a(G') + \ell + 1$. By assumption, we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + \ell + 1 \leq a(G') + \ell + 2 \leq a(G) + 1.$$

□

Lemma 9 *Let H and P be two vertex-disjoint graphs, where P is the paw graph and $n(H) \geq 3$. Moreover, let z be a vertex of H and let x be a vertex of P with $d_P(x) = 2$. Assume that G is the graph obtained from H and P by identifying z and x. Then, there exists a connected graph G' with $f(G') < f(G)$ such that $\gamma_t(G') \leq a(G') + 1$ implies $\gamma_t(G) \leq a(G) + 1$.*

Proof. We denote the neighbors of x in P with u and w, and let v be the leaf neighbor of u. Two subcases will be considered depending on the degree $d(x)$ of x in G.

First suppose that $d(x) = 3$. Denote the third neighbor of x outside P with y. Since H had at least three vertices, y is not a leaf, and hence $G' = G - V(P) = G - \{x, u, v, w\}$ is not a trivial graph. Also, $m(G') = m(G) - 5$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, w\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. If S' is an optimal annihilation set of G', we have $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 4$. Let $S = S' \cup \{u, v\}$. Then $\sum(S, G) = \sum(S', G) + d(u) + d(v) \leq m(G) - 4 + 3 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. Then, $\gamma_t(G') \leq a(G') + 1$ implies

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1. \quad (3)$$

Now, suppose $d(x) \geq 4$. In this case let $G' = G - \{u, v, w\}$, and so $m(G') = m(G) - 4$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, w\}$ is a total dominating set of graph G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. Now, let S' be an optimal annihilation set in G'. If $x \notin S'$, then $\sum(S', G) = \sum(S', G')$. In this case, let $S = S' \cup \{u, v\}$. Then $\sum(S, G) = \sum(S', G) + d(u) + d(v) \leq m(G) - 4 + 3 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. If $\gamma_t(G') \leq a(G') + 1$, the chain (3) of inequalities verifies the statement.

But, if $x \in S'$, then $\sum(S', G) = \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 4 + 2 = m(G) - 2$. In this case, let $S = (S' \backslash \{x\}) \cup \{u, v, w\}$. Since $d(x) \geq 4$ we have $\sum(S, G) = \sum(S', G) - d(x) + 3 + 1 + 2 \leq m(G) - 2 - 4 + 6 = m(G)$, implying that $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. By assumption we have that $\gamma_t(G') \leq a(G') + 1$. Therefore, we get again (3) which proves the lemma. □

3 Cactus graphs

Recall that a cactus graph is a connected graph such that any two of its cycles are pairwise edge-disjoint. If the cactus graph does not contain any cycles, then it is a tree. Let C^1 and C^2 be two cycles in the cactus graph. We define

$$d\left(C^1, C^2\right) = \min\left\{d(u, v) \mid u \in V\left(C^1\right), v \in V\left(C^2\right)\right\}$$

where $d(u, v)$ denotes the distance between vertices u and v. Let $x_1 \in V\left(C^1\right)$ and $x_2 \in V\left(C^2\right)$ be two vertices such that $d(x_1, x_2) = d\left(C^1, C^2\right)$. Then we call x_1 and x_2 *exit-vertices* of cycles C^1 and C^2, respectively. A cycle is said to be an *outer cycle* if it has at most one exit-vertex. If a cactus graph is not a tree, then by the definition of a cactus graph it must contain at least one outer cycle. Note that a cactus graph, which is neither a tree nor a cycle, does not contain exit-vertices if and only if it is unicyclic. In this case, we will take an arbitrary vertex of the unique cycle whose degree is at least 3 for the role of the exit-vertex x. In the right-hand side graph of Figure 2, we have three possibilities for the choice of that vertex x (either x_1 or x_2 or x_3). In both cases, whether a cactus graph has one or more cycles, vertex x will always have degree $d(x) \geq 3$.

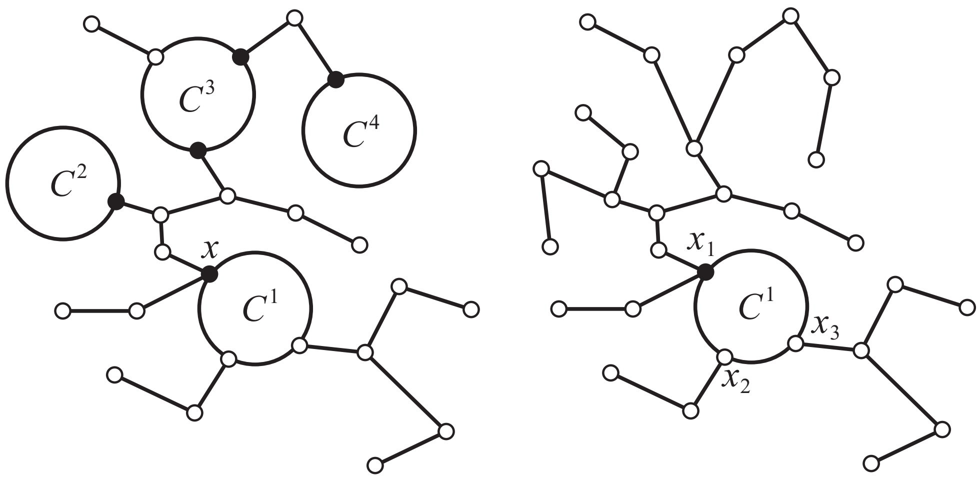

Figure 2: Two examples of cactus graphs. The first one has three outer cycles (C^1, C^2, C^4), its exit-vertices are filled with black. The second cactus graph is unicyclic with one outer cycle, and has no exit-vertices.

In this section we prove Conjecture 1 for cactus graphs. Recall the corresponding statement.

Theorem 3. *If G is a nontrivial cactus graph, then $\gamma_t(G) \leq a(G) + 1$.*

Proof. We proceed by induction on the value of function $f(G) \geq 7$. For $f(G) = 7$ we have $G \cong K_2$, and $\gamma_t(K_2) = 2 = a(K_2)+1$. For the inductive hypothesis, let $f(G) \geq 8$ and assume that for every nontrivial cactus graph G' with $f(G') < f(G)$ we have $\gamma_t(G') \leq a(G') + 1$. If G is a tree, then by Theorem 2 the result follows. Also, if G is a cycle, the statement is true. Thus, we my suppose that G contains at least one cycle as a proper subgraph. We denote with C_k, $k \geq 3$, an outer cycle of G.

Through most part of the proof, we will consider cactus graphs G' formed from G by removing a set of vertices in such a way that graph G' will still be a connected cactus graph and consequently $f(G') < f(G)$ will hold. Throughout, S' will denote an optimal annihilation set in G'. We consider two cases.

Case 1: All vertices from $V(C_k)\backslash\{x\}$ have degree 2.
Lemma 6(ii) and our inductive hypothesis together imply that $\gamma_t(G) \leq a(G) + 1$.

Case 2: There exists a vertex from $V(C_k)\backslash\{x\}$ that has degree at least 3.
Since $V(C_k)\backslash\{x\}$ contains some vertices of degree at least 3, and C_k is an outer cycle, there are trees attached to those vertices. Suppose, we root all trees in the vertices $V(C_k)\backslash\{x\}$ to which these trees are attached. Amongst those trees we consider the tree T with the largest height $h(T) = \max\{d(u,v) \,|\, u = V(C_k) \cap V(T), v \in V(T)\}$. Denote this maximum height with $h \geq 1$ and let u be the vertex of $V(C_k)\backslash\{x\}$ to which tree T is attached. We consider three subcases.

Case 2.1: $h \geq 3$.
Since $h \geq 3$, there exists a leaf $v \in V(T)$ such that $d(u,v) = h \geq 3$. By Lemma 7 and our inductive hypothesis, graph G satisfies (1).

Case 2.2: $h = 2$.
We only need to consider the four cases shown in Figure 3. All other cases for $h = 2$ can be proved with the help of Lemma 5(ii) and (v).

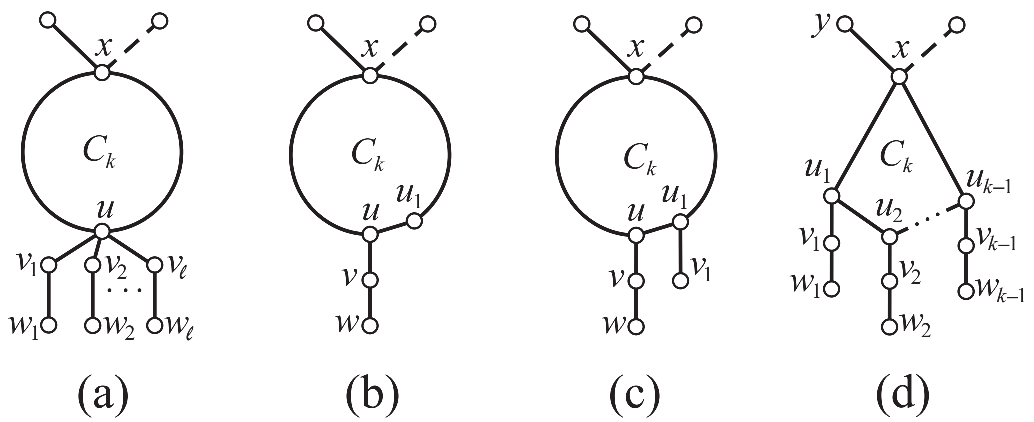

Figure 3: Cases for $h = 2$

We first start with the case in Figure 3(a). In this case, we have a subdivided star $K_{1,\ell}$, $\ell \geq 2$, attached to the outer cycle, and hence, by Lemma 8 and our inductive hypothesis for $G' = G - V(S(K_{1,\ell}))$, graph G satisfies (1).

Next, we consider the case in Figure 3(b). Vertex u has only one path of length 2 attached to it, i.e. $d(u) = 3$. We suppose that u has a neighbor u_1 in $V(C_k)\backslash\{x\}$ with degree

$d(u_1) = 2$. We denote with v the only child of u, and with w the only child of v. Let $G' = G - \{u, u_1, v, w\}$, and so $m(G') = m(G) - 5$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, v\}$ is a total dominating set of graph G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. Independently of whether the neighbors of u and u_1 in G' are inside S' or not, we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 3$. Let $S = S' \cup \{v, w\}$. Then $\sum(S, G) = \sum(S', G) + d(v) + d(w) \leq m(G) - 3 + 2 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

We proceed with the case in Figure 3(c). Vertex u has again only one path of length 2 attached to it, i.e. $d(u) = 3$. We suppose that u has a neighbor u_1 in $V(C_k) \backslash \{x\}$ with degree $d(u_1) = 3$, and a path of length 1 attached to it. Denote its child with v_1. We also denote with v the only child of u, and with w the only child of v. Let $G' = G - \{u, v, w, u_1, v_1\}$, and so $m(G') = m(G) - 6$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, v, u_1\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 3 = \gamma_t(G') + 3$. Independently of whether the neighbors of u and u_1 in G' are inside S' or not, we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 4$. Let $S = S' \cup \{v, w, v_1\}$. Then $\sum(S, G) = \sum(S', G) + d(v) + d(w) + d(v_1) \leq m(G) - 4 + 2 + 1 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 3 = a(G') + 3$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 3 \leq a(G') + 4 \leq a(G) + 1.$$

The last case to consider is the one shown in Figure 3(d). Denote with $u_1, \ldots, u_{k-1}$ all vertices of $V(C) \backslash \{x\}$. Each of those vertices must have one path of length 2 attached to it, i.e. $d(u_i) = 3$ for every $i \in \{1, \ldots, k-1\}$, since otherwise this case would be covered by one of the previous three cases. Clearly, vertices u_1 and u_{k-1} are neighbors of x. Denote for every $i \in \{1, \ldots, k-1\}$ with v_i the only child of u_i, and with w_i the only child of v_i. We consider two subcases.

First, suppose that $d(x) = 3$. Denote the third neighbor of x outside C_k with y. If vertex y was a leaf, then we could exchange vertex x with one of u_i's, and use the proof for the case in Figure 3(c). Hence, we may assume that y is not a leaf and graph $G' = G - \{x, u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$ is not a trivial cactus graph. Also, $m(G') = m(G) - 3k + 1$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2k - 2 = \gamma_t(G') + 2k - 2$. Independently of whether x is inside S' or not we have $\sum(S', G) \leq \sum(S', G') + 1 \leq m(G') + 1 = m(G) - 3k + 2$. Let $S = S' \cup \{v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$. Then $\sum(S, G) = \sum(S', G) + 2(k-1) + (k-1) \leq m(G) - 3k + 2 + (3k - 3) = m(G) - 1$, and we have $a(G) \geq |S| = |S'| + 2k - 2 = a(G') + 2k - 2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2k - 2 \leq a(G') + 2k - 1 \leq a(G) + 1.$$

Now, suppose that $d(x) \geq 4$. Let $G' = G - \{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$, and so $m(G') = m(G) - 3k + 2$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, \ldots, u_{k-1}, v_1, \ldots, v_{k-1}\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq$

$|D| = |D'| + 2k - 2 = \gamma_t(G') + 2k - 2$. If $x \notin S'$, then $\sum(S', G) = \sum(S', G')$. In this case, let $S = S' \cup \{v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$. Then $\sum(S, G) = \sum(S', G) + 2(k-1) + (k-1) \leq m(G) - 1$, and we have $a(G) \geq |S| = |S'| + 2k - 2 = a(G') + 2k - 2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2k - 2 \leq a(G') + 2k - 1 \leq a(G) + 1.$$

If $x \in S'$, then $\sum(S', G) = \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 3k + 2 + 2 = m(G) - 3k + 4$. In this case, let $S = (S' \backslash \{x\}) \cup \{u_1, v_1, \ldots, v_{k-1}, w_1, \ldots, w_{k-1}\}$. Since $d(x) \geq 4$ we have $\sum(S, G) = \sum(S', G) - d(x) + d(u_1) + 2(k-1) + (k-1) \leq m(G) - 3k + 4 - 4 + 3 + 3(k-1) = m(G)$, implying that $a(G) \geq |S| = |S'| + 2k - 2 = a(G') + 2k - 2$. By our inductive hypothesis, we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2k - 2 \leq a(G') + 2k - 1 \leq a(G) + 1.$$

Case 2.3: $h = 1$.
It suffices to consider only those cases shown in Figure 4. Note that all other cactus graphs with $h = 1$ would involve two leaves at distance of at most 3, and hence these cases can be reduced to the direct application of Lemma 5(ii) and (v).

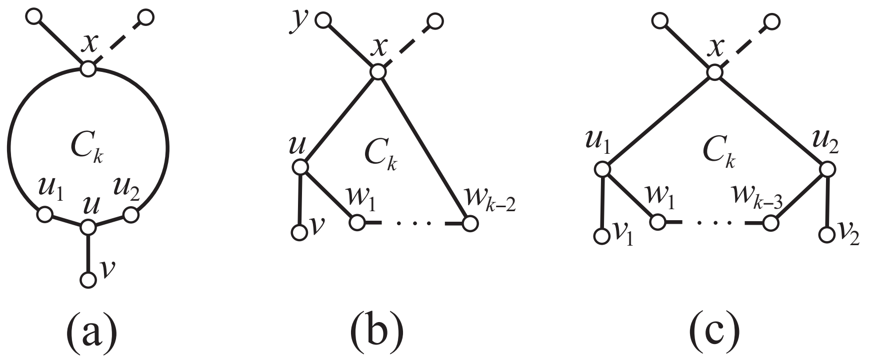

Figure 4: Cases for $h = 1$

First, consider Figure 4(a). Here, we assume that vertex u has degree $d(u) = 3$, and its neighbors in $V(C_k) \backslash \{x\}$, namely u_1 and u_2, are of degree 2. Denote the child of u with v. In this case we want u_1 and u_2 to be different from the exit-vertex x. Let $G' = G - \{u, v, u_1, u_2\}$, and so $m(G') = m(G) - 5$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u, u_1\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 2 = \gamma_t(G') + 2$. Independently of whether the neighbors of u_1 and u_2 in G' are inside S' or not, we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 3$. Let $S = S' \cup \{u, v\}$. Then $\sum(S, G) = \sum(S', G) + d(u) + d(v) \leq m(G) - 3 + 2 + 1 = m(G)$, and we have $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

We proceed with the case in Figure 4(b). Denote with u the vertex of $V(C_k) \backslash \{x\}$ with one path of length 1 attached to it, i.e. $d(u) = 3$, and let v be its only child. One of the neighbors of u must clearly be vertex x because otherwise we would have the case in Figure 4(a). Suppose that all other vertices in $V(C_k) \backslash \{x\}$, denote them with $w_1, \ldots, w_{k-2}$, have degree 2.

First suppose that $k=3$. In this case x, u, v and w induce the paw graph. Denote a neighbor of x outside this paw with y. If y is a leaf, then G is graph formed from a cycle C_3 on vertices x, u, w_1 with two leaves v and y adjacent to u and x, respectively. For such graph G we have $\gamma_t(G)=2$ and $a(G)=3$. Thus, (1) holds for G. So we may assume that y is not a leaf. Then, by Lemma 9 and our inductive hypothesis, graph G satisfies (1).

Let $k=4$. Let $G'=G-\{u,v,w_1,w_2\}$, and so $m(G')=m(G)-5$. If D' is a minimum total dominating set of G', then $D=D'\cup\{x,u\}$ is a total dominating set of G, and hence $\gamma_t(G)\leq|D|=|D'|+2=\gamma_t(G')+2$. Independently of whether x is inside S' or not we have $\sum(S',G)\leq\sum(S',G')+2\leq m(G')+2=m(G)-3$. Let $S=S'\cup\{v,w_1\}$. Then $\sum(S,G)=\sum(S',G)+d(v)+d(w_1)\leq m(G)-3+1+2=m(G)$, and we have $a(G)\geq|S|=|S'|+2=a(G')+2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G')\leq a(G')+1$. Therefore,

$$\gamma_t(G)\leq\gamma_t(G')+2\leq a(G')+3\leq a(G)+1.$$

Now, let $k=5$. We have three consecutive degree-2 vertices w_1, w_2, w_3, and there is an edge between u and x, which are neighbors of w_1 and w_3, respectively. Let $G'=G-\{u,v,w_1,w_2,w_3\}$, and so $m(G')=m(G)-6$. If D' is a minimum total dominating set of G', then $D=D'\cup\{x,u,w_3\}$ is a total dominating set of G, and hence $\gamma_t(G)\leq|D|=|D'|+3=\gamma_t(G')+3$. If $x\notin S'$, then $\sum(S',G)=\sum(S',G')\leq m(G)-6$. In this case, let $S=S'\cup\{v,w_1,w_2\}$. Then $\sum(S,G)=\sum(S',G)+d(v)+d(w_1)+d(w_2)\leq m(G)-6+1+2+2=m(G)-1$, and we have $a(G)\geq|S|=|S'|+3=a(G')+3$. Applying our inductive hypothesis to G', we have that $\gamma_t(G')\leq a(G')+1$. Therefore,

$$\gamma_t(G)\leq\gamma_t(G')+3\leq a(G')+4\leq a(G)+1.$$

But if $x\in S'$, then $\sum(S',G)=\sum(S',G')+2\leq m(G)-6+2=m(G)-4$. In this case, let $S=(S'\backslash\{x\})\cup\{v,w_1,w_2,w_3\}$. Since $d(x)\geq 3$, we have $\sum(S,G)=\sum(S',G)-d(x)+d(v)+d(w_1)+d(w_2)+d(w_3)\leq m(G)$, which implies $a(G)\geq|S|=|S'|+3=a(G')+3$. Applying our inductive hypothesis to G', we again have that $\gamma_t(G')\leq a(G')+1$. Therefore,

$$\gamma_t(G)\leq\gamma_t(G')+3\leq a(G')+4\leq a(G)+1.$$

For the last case, let $k\geq 6$. Now we have three consecutive vertices w_1, w_2, w_3 with degree $d(w_1)=d(w_2)=d(w_3)=2$, and there is no edge between u and w_4. Thus, Lemma 5(iii) and our inductive hypothesis imply that graph G satisfies (1).

We finish with the case in Figure 4(c). Denote with u_1 and u_2 two vertices in $V(C_k)\backslash\{x\}$ each with one path of length 1 attached to it, i.e. $d(u_1)=d(u_2)=3$, and let v_1 and v_2 be the only child of u_1 and u_2, respectively. The exit-vertex x must be the neighbor of both u_1 and u_2 because otherwise we would have the case in Figure 4(a). We denote all vertices in $V(C_k)\backslash\{x\}$ between vertex u_1 and u_2 with $w_1,\ldots,w_{k-3}$. Those vertices have all degree 2.

If $k=3$, the statement follows immediately from the hypothesis and Lemma 5(v), since in this case the support vertices of v_1 and v_2 are adjacent. Thus, we first suppose that $k=4$. Let $G'=G-\{u_1,v_1,u_2,v_2,w_1\}$, and so $m(G')=m(G)-6$. If D' is a minimum total dominating set of G', then $D=D'\cup\{x,u_1,u_2\}$ is a total dominating set of G, and hence $\gamma_t(G)\leq|D|=|D'|+3=\gamma_t(G')+3$. Independently of whether $x\in S'$ or $x\notin S'$, we have $\sum(S',G)\leq\sum(S',G')+2\leq m(G')+2=m(G)-4$. Let $S=S'\cup\{v_1,v_2,w_1\}$. Then $\sum(S,G)=\sum(S',G)+d(v_1)+d(v_2)+d(w_1)\leq m(G)-4+1+1+2=m(G)$, and we have

$a(G) \geq |S| = |S'| + 3 = a(G') + 3$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 3 \leq a(G') + 4 \leq a(G) + 1.$$

Now, suppose that $k = 5$. We make a similar cut than the one for $k = 4$. Let $G' = G - \{u_1, v_1, u_2, v_2, w_1, w_2\}$, and so $m(G') = m(G) - 7$. If D' is a minimum total dominating set of G', then $D = D' \cup \{x, u_1, u_2\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'| + 3 = \gamma_t(G') + 3$. For any optimal annihilation set S' of G', we have $\sum(S', G) \leq \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 5$. Let $S = S' \cup \{v_1, v_2, w_1\}$. Then $\sum(S, G) = \sum(S', G) + d(v_1) + d(v_2) + d(w_1) \leq m(G) - 1$, and $a(G) \geq |S| = |S'| + 3 = a(G') + 3$ follows. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G') + 1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G') + 3 \leq a(G') + 4 \leq a(G) + 1.$$

For the last case, let $k \geq 6$. We have again three consecutive vertices w_1, w_2, w_3 with degree $d(w_1) = d(w_2) = d(w_3) = 2$. Furthermore, vertices u_1 and u_2 are not adjacent. Thus, by Lemma 5(iii) and our inductive hypothesis, graph G satisfies (1).

These cover all possible cases which can occur in a cactus graph which is neither a tree nor a cycle. Hence, Conjecture 1 is true for the family of cactus graphs.

□

4 Block graphs

Recall that a block graph is a connected graph in which every 2-connected component (block) is a clique. Block graphs have minimum degree at least 3 if its building blocks are complete graphs K_k, $k \geq 4$. Thus, Conjecture 1 obviously holds for them. On the other hand, block graphs also contain blocks K_2 and K_3, and therefore, it clearly makes sense to study Conjecture 1 on block graphs.

We proceed with a similar definition than the one for cactus graphs. If all cliques in a block graph are K_2, then it is a tree. For every $k \geq 3$ we will call complete graph K_k a *complex clique*. Let K^1 and K^2 be two complex cliques in the block graph. We define

$$d\left(K^1, K^2\right) = \min\left\{d(u, v) \mid u \in V\left(K^1\right), v \in V\left(K^2\right)\right\}$$

where $d(u, v)$ denotes the distance between vertices u and v. Let $x_1 \in V\left(K^1\right)$ and $x_2 \in V\left(K^2\right)$ be two vertices such that $d(x_1, x_2) = d(K^1, K^2)$. Then we call x_1 and x_2 exit-vertices of complex cliques K^1 and K^2, respectively. Notice, that a complex clique might not have any exit-vertices if it is the only complex clique in the block graph. A complex clique will be called an *outer complex clique* if it has at most one exit-vertex. If a block graph is not a tree, then by the definition of a block graph it must contain at least one outer complex clique.

Now, we are ready to present a proof of Theorem 4. Recall its statement.

Theorem 4. *If G is a nontrivial block graph, then $\gamma_t(G) \leq a(G) + 1$.*

Proof. We proceed by induction on the value of function $f(G)$. For $f(G) = 7$ we have $G \cong K_2$, and $\gamma_t(K_2) = 2 = a(K_2) + 1$. For the inductive hypothesis, let $f(G) \geq 8$ and assume that for every nontrivial block graph G' with $f(G') < f(G)$ we have $\gamma_t(G') \leq a(G') + 1$. If G does not contain complex cliques, then it is a tree, and by Theorem 2 the result follows. Also, if G is a clique, i.e. $G \cong K_\ell$, $\ell \geq 2$, we have $\gamma_t(K_\ell) = 2 \leq a(K_\ell) + 1$. Thus, we my suppose that G is neither a tree nor a clique, but contains at least one complex clique as a proper subgraph. We denote with K_k an outer complex clique of G. Similarly than in the proof for cactus graphs, all outer complex cliques in the figures will be drawn with an exit-vertex x even though a unique complex clique in a block graph does not have one. In the latter case, we denote with x an arbitrary vertex of clique K_k whose degree is at least k. In both cases, whether a block graph has one or more complex cliques, vertex x will have degree $d(x) \geq k$.

Through most part of the proof, we will consider block graphs G' formed from G by removing a set of vertices in such a way that graph G' will still be a connected block graph and consequently $f(G') < f(G)$ will hold. Throughout, S' will denote an optimal annihilation set in G'. We consider two cases.

Case 1: All vertices from $V(K_k)\backslash\{x\}$ have degree $k-1$.
Let $u_1, \ldots, u_{k-1}$ be vertices from $V(K_k)\backslash\{x\}$ with degree $k-1$. By Lemma 6(i), and inductive hypothesis for $G' = G - \{u_1, \ldots, u_{k-1}\}$, graph G satisfies (1).

Case 2: There exists a vertex from $V(K_k)\backslash\{x\}$ that has degree at least k.
Since $V(K_k)\backslash\{x\}$ contains vertices of degree at least k, and K_k is an outer complex clique, there are trees attached to those vertices. Suppose, we root all trees in the vertices $V(K_k)\backslash\{x\}$ to which these trees are attached. Amongst those threes we consider the tree T with the largest height h. Let u be the vertex of $V(K_k)\backslash\{x\}$ to which this tree T is attached. We split the problem into three subcases.

Case 2.1: $h \geq 3$.
Since $h \geq 3$, there exists a leaf $v \in V(T)$ such that $d(u, v) = d \geq 3$. By Lemma 7 and our inductive hypothesis, graph G satisfies (1).

Case 2.2: $h = 2$.
We only need to consider cases shown in Figure 5. All other cases for $h = 2$ can be proved with the help of Lemma 5(ii) and (v).

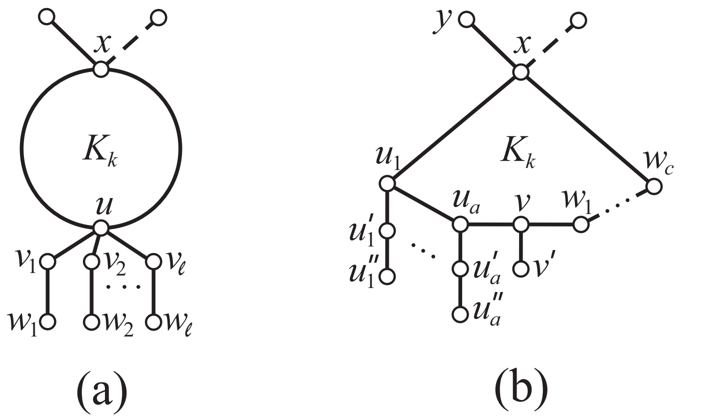

Figure 5: Cases for $h = 2$

We start with the case in Figure 5(a) and suppose that there exists a subdivided star $K_{1,\ell}$, $\ell \geq 2$, attached to the outer complex clique. By Lemma 8 and our inductive hypothesis for $G' = G - V(S(K_{1,\ell}))$, graph G satisfies (1).

In the case shown in Figure 5(b), there are vertices in $V(K_k)\backslash\{x\}$ such that a path of length 2 is attached to them. We denote such vertices with $u_1, \ldots, u_a$. Since $h = 2$, we must have at least one such vertex. Thus, $a \in \{1, \ldots, k-1\}$. For each $i \in \{1, \ldots, a\}$ we denote with u_i' the child of u_i, and with u_i'' the child of u_i'. Also, we may suppose that at most one vertex in $V(K_k)\backslash\{x\}$ has a path of length 1 attached to it. If we had more such vertices, then we would have two adjacent support vertices and we could prove the statement by referring to Lemma 5(v). Hence, denote this vertex with v and let b denote the Boolean value whether it exists in $V(K_k)\backslash\{x\}$ or not, i.e. $b \in \{0, 1\}$. We denote the child of v with v'. There may also be some vertices in $V(K_k)\backslash\{x\}$ without a path attached to them. Denote them with $w_1, \ldots, w_c$, $c \in \{0, \ldots, k-2\}$. Clearly, we have $a + b + c = k - 1$. Let $G' = G - \{u_1, \ldots u_a, u_1', \ldots u_a', u_1'', \ldots, u_a'', v, v', w_1, \ldots, w_c\}$, and so $m(G') = m(G) - \left(\frac{k(k-1)}{2} + 2a + b\right)$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, \ldots, u_a, u_1', \ldots, u_a', v\}$ is a total dominating set of G, and hence $\gamma_t(G) \le |D| = |D'| + 2a + b = \gamma_t(G') + 2a + b$. Independently of whether x is inside S' or not we have $\sum(S', G) \le \sum(S', G') + (k-1) = m(G') + (k-1) = m(G) - \left(\frac{k(k-1)}{2} + 2a + b\right) + k - 1$. Let $S = S' \cup \{u_1', \ldots, u_a', u_1'', \ldots u_a'', v'\}$. Suppose that $1 \le a \le k-2$ holds. Then, the last one from the following inequalities is valid and it implies the first one:

$$\begin{aligned} -\left(\frac{k(k-1)}{2} + 2a + b\right) + k - 1 + 3a + b &\le 0 \\ k^2 - 3k - 2a + 2 &\ge 0 \\ k^2 - 3k - 2(k-2) + 2 &\ge 0 \\ k^2 - 5k + 6 &\ge 0. \end{aligned}$$

Similarly, under the conditions $a = k - 1 \ge 3$, the following relations hold:

$$\begin{aligned} -\left(\frac{k(k-1)}{2} + 2a + b\right) + k - 1 + 3a + b &\le 0 \\ k^2 - 3k - 2(k-1) + 2 &\ge 0 \\ k^2 - 5k + 4 &\ge 0. \end{aligned}$$

In both cases we get $\sum(S, G) = \sum(S', G) + d(u_1') + \ldots + d(u_a') + d(u_1'') + \ldots + d(u_a'') + d(v') \le m(G) - \left(\frac{k(k-1)}{2} + 2a + b\right) + k - 1 + 3a + b \le m(G)$, which implies $a(G) \ge |S| = |S'| + 2a + b = a(G') + 2a + b$. By our inductive hypothesis, G' satisfies Conjecture 1. Consequently,

$$\gamma_t(G) \le \gamma_t(G') + 2a + b \le a(G') + 2a + b + 1 \le a(G) + 1.$$

What remains is to establish the statement for $k = 3$ and $a = k - 1 = 2$. We consider two subcases. First, suppose that $d(x) = 3$. Denote the third neighbor of x outside K_3 with y. If vertex y was a leaf, then we could exchange vertex x either with u_1 or u_2, and apply the proof for the case $a = 1 = k - 2$. Hence, we may assume that y is not a leaf, and therefore, graph $G' = G - \{x, u_1, u_1', u_1'', u_2, u_2', u_2''\}$ is not a trivial block graph. Also, $m(G') = m(G) - 8$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, u_1', u_2, u_2'\}$ is a total dominating set of graph G, and hence $\gamma_t(G) \le |D| = |D'| + 4 = \gamma_t(G') + 4$. Independently of whether y is inside S' or not we have $\sum(S',G) \leq \sum(S',G')+1 \leq m(G')+1 = m(G)-8+1 = m(G)-7$. Let $S = S' \cup \{u_1', u_1'', u_2', u_2''\}$. Then $\sum(S,G) = \sum(S',G) + d(u_1') + d(u_\ell'') + d(u_2') + d(u_2'') \leq m(G)-7+2+1+2+1 = m(G)-1$, and we have $a(G) \geq |S| = |S'|+4 = a(G')+4$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G')+1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G')+4 \leq a(G')+5 \leq a(G)+1.$$

Now, suppose that $d(x) \geq 4$. Let $G' = G - \{u_1, u_1', u_1'', u_2, u_2', u_2''\}$, and so $m(G') = m(G)-7$. If D' is a minimum total dominating set of G', then $D = D' \cup \{u_1, u_1', u_2, u_2'\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'|+4 = \gamma_t(G)+4$. If $x \notin S'$, then $\sum(S',G) = \sum(S',G')$. In this case, let $S = S' \cup \{u_1', u_1'', u_2', u_2''\}$. Then $\sum(S,G) = \sum(S',G) + d(u_1') + d(u_1'') + d(u_2') + d(u_2'') \leq m(G)-7+2+1+2+1 = m(G)-1$, and we have $a(G) \geq |S| = |S'|+4 = a(G')+4$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G')+1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G')+4 \leq a(G')+5 \leq a(G)+1.$$

If $x \in S'$, then $\sum(S',G) = \sum(S',G')+2 \leq m(G')+2 = m(G)-7+2 = m(G)-5$. In this case, let $S = (S' \backslash \{x\}) \cup \{u_1, u_1', u_1'', u_2', u_2''\}$. Since $d(x) \geq 4$ we have $\sum(S,G) = \sum(S',G) - d(x) + d(u_1) + d(u_1') + d(u_1'') + d(u_2') + d(u_2'') \leq m(G)-5-4+3+2+1+2+1 = m(G)$, implying that $a(G) \geq |S| = |S'|+4 = a(G')+4$. Applying again our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G')+1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G')+4 \leq a(G')+5 \leq a(G)+1.$$

Case 2.3: $h = 1$.
We need to consider only one case which is shown in Figure 6. Again, all other cases for $h = 1$ can be proved with the help of Lemma 5(ii) and (v).

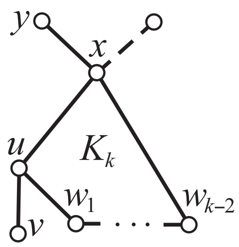

Figure 6: The case for $h = 1$

We may also suppose that there is at most one vertex in $V(K_k)\backslash\{x\}$ which has a path of length 1 attached to it. If we had more such vertices, then we would have adjacent support vertices and we could prove this case with Lemma 5(v). Denote this vertex with u and its child with v. There are also vertices in $V(K_k)\backslash\{x\}$ without a path attached to them. Denote them with $w_1, \ldots, w_{k-2}$. Let $G' = G - \{u, v, w_1, \ldots, w_{k-2}\}$, and so $m(G') = m(G) - \left(\frac{k(k-1)}{2}+1\right)$. If D' is a minimum total dominating set of G', then $D = D' \cup \{x, u\}$ is a total dominating set of G, and hence $\gamma_t(G) \leq |D| = |D'|+2 = \gamma_t(G')+2$. Independently of whether x is inside S' or not we have $\sum(S',G) \leq \sum(S',G')+k-1 = m(G')+k-1 = m(G) - \left(\frac{k(k-1)}{2}+1\right)+k-1$.

Let $S = S' \cup \{v, w_1\}$. For $k \geq 4$ we have:

$$\begin{aligned} -\left(\frac{k(k-1)}{2}+1\right)+k-1+1+k-1 &\leq 0 \\ k^2-k-4k+4 &\geq 0 \\ k^2-5k+4 &\geq 0. \end{aligned}$$

It follows that $\sum(S,G) = \sum(S',G)+d(v)+d(w_1) \leq m(G)-\left(\frac{k(k-1)}{2}+1\right)+k-1+1+k-1 \leq m(G)$, which implies $a(G) \geq |S| = |S'|+2 = a(G')+2$. Applying our inductive hypothesis to G', we have that $\gamma_t(G') \leq a(G')+1$. Therefore,

$$\gamma_t(G) \leq \gamma_t(G')+2 \leq a(G')+3 \leq a(G)+1.$$

We end the proof with $k=3$. In this case, x, u, v and w_1 induce the paw graph. Denote the third neighbor of x outside K_3 with y. If y is a leaf, then G is graph formed from a clique K_3 on vertices x, u, w_1 with two leaves v and y adjacent to u and x, respectively. For such graph G we have $\gamma_t(G) = 2$ and $a(G) = 3$. Thus, (1) holds for G. So we may assume that y is not a leaf. Then, by Lemma 9 and our inductive hypothesis, graph G satisfies (1).

We have considered all possible cases which can occur in a block graph which is neither a tree nor a clique. Hence, Conjecture 1 is true over the family of block graphs. □

5 Concluding remarks

To show that our main results, namely Theorem 3 and 4, are sharp, we remark that trees are included in both classes. Therefore, we may refer to the family of trees characterized in [4] which satisfy Conjecture 1 with equality.

We may also observe that even cycles C_n, where $n \equiv 2 \pmod 4$, have $\gamma_t(C_n) = \frac{n}{2}+1$ and $a(C_n) = \frac{n}{2}$. Also, there are arbitrarily large cactus graphs which are neither trees nor cycles, but satisfy $\gamma_t(G) = a(G)+1$. For an integer $k \geq 2$, take the vertex-disjoint cycles $G_i \cong C_6$, $1 \leq i \leq k$, and for each $i \in \{1, \dots, k-1\}$ identify one vertex from $V(G_i)$ and one from $V(G_{i+1})$ such that the constructed graph G satisfies $\Delta(G) \leq 4$. One can check that $m(G) = 6k$; $\gamma_t(G) = 3k+1$; and $a = 3k$. Thus, Theorem 3 holds with equality for the graphs G constructed this way. The following characterization problem remains open.

Problem 10 *Characterize cactus graphs which satisfy $\gamma_t(G) = a(G)+1$.*

For block graphs, already the following question might be interesting.

Problem 11 *Does there exist a block graph G which is neither a tree nor a K_3 but satisfies $\gamma_t(G) = a(G)+1$?*

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