Relating the annihilation number and the 2-domination number of block graphs

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Abstract

The 2-domination number $\gamma_2(G)$ of a graph G is the order of a smallest set $D \subseteq V(G)$ such that each vertex of $V(G) \backslash D$ is adjacent to at least two vertices in D. The annihilation number $a(G)$ of G is the largest integer k such that there exist k different vertices in G with degree sum of at most $|E(G)|$. It is conjectured that $\gamma_2(G) \leq a(G) + 1$ holds for every nontrivial connected graph G. The conjecture was proved for graphs with minimum degree at least 3, and remains unresolved for graphs with minimum degree 1 or 2. In this paper we establish the conjecture for block graphs.

Key words: 2-domination number, Annihilation number, Block graph.

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1 Introduction

Graphs considered in this paper are nontrivial, finite, simple, and undirected. A nontrivial graph means a graph on at least two vertices. Let $G = (V, E)$ be a graph. Then $V = V(G)$ denotes the set of vertices of order $n(G) = |V|$, and $E = E(G)$ denotes the set of edges of size $m(G) = |E|$. The *degree* of a vertex $v \in V$ in graph G is the number of edges incident with vertex v and is denoted by $d_G(v)$. A vertex v of degree 1 is called a *pendant vertex* (or a leaf), while its only neighbor is called a *support vertex*. If a vertex has at least two neighbors which are pendant vertices, then we refer to it as a *strong support vertex*. The minimum and maximum degree among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a vertex $v \in V(G)$, the set of its neighbors is called the *open neighborhood of* v and denoted by $N_G(v)$. If G is clear from the context, then instead of $d_G(v)$ and $N_G(v)$ we will write $d(v)$ and $N(v)$, respectively.

For a set $X \subseteq V$ we define the graph $G - X$ as the graph obtained from G by deleting the vertices in X and all edges incident with them. Moreover, if $u_1u_2 \in E$ and $v_1v_2 \notin E$, where

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$u_1, u_2, v_1, v_2 \in V$, notations $G - u_1u_2$ and $G + v_1v_2$ will be used for the graphs $(V, E \setminus \{u_1u_2\})$ and $(V, E \cup \{v_1v_2\})$, respectively.

A connected graph is called a *block graph* if each 2-connected component of G is a clique. An example of a block graph can be seen in Figure 2.

For $k \geq 1$, a *k-dominating set* of a graph G is a set $D \subseteq V(G)$ such that each vertex of $V(G)\backslash D$ is adjacent to at least k vertices in D. The *k-domination number* of G is the minimum cardinality of a k-dominating set of G, and is denoted by $\gamma_k(G)$. Thus, a 1-dominating set is a usual dominating set, and the 1-dominating number equals the well-known and studied domination number $\gamma(G)$. The notion of a k-dominating set was introduced by Fink and Jacobson [15]. More on k-domination can be found in the book written by Haynes, Hedetniemi, and Slater [19], and a nice survey was written on this topic by Chellali, Favaron, Hansberg, and Volkamnn [7].

Domination has applications in sensor networks which can be modeled as graphs such that the vertices represent the sensors and two vertices are adjacent if and only if the corresponding devices can communicate with each other. Then, a dominating set D of such a graph G can be interpreted as a collection of cluster-heads, as each sensor which does not belong to D has at least one head within communication distance. In this sense, a k-dominating set D may represent a dominating set which is $(k-1)$-fault tolerant, which means that in the case of failure of at most $(k-1)$ cluster-heads, each remaining vertex is either in connection with at least one head or is a head by itself. The price of this k-fault tolerance can be very high, but for the usual cases arising in practice, 2-domination might be enough and it does not require extremely many heads. Since it is important to lower the cost of sensors, it is necessary to obtain good bounds for this invariant. Therefore, the concept of 2-domination was studied extensively. Good bounds for k-domination number, and in particular 2-domination number, were obtained in [5, 6, 13, 14, 17, 18]. There are many further kinds of applications of k-domination, e.g. in [20] a facility location problem was described, and requires that each region is either served by its own facility or has at least two neighboring regions with such a service. Recently, different versions of 2-domination were considered, for instance 2-rainbow domination and Roman 2-domination [1, 2, 8, 9].

In this paper we focus on the 2-domination number and its relation to another invariant called the annihilation number. For a subset $S \subseteq V(G)$ we define

$$\sum(S, G) = \sum_{v \in S} d_G(v).$$

Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G such that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$. The *annihilation number* $a(G)$ is the largest integer k such that $\sum_{i=1}^{k} d(v_i) \leq m(G)$, or equivalently, the largest integer k such that $\sum_{i=1}^{k} d(v_i) \leq \sum_{i=k+1}^{n} d(v_i)$. The value $a = a(G)$ is the only integer satisfying both

$$\sum_{i=1}^{a} d(v_i) \leq m(G) \quad \text{and} \quad \sum_{i=1}^{a+1} d(v_i) \geq m(G) + 1.$$

The annihilation number was first introduced by Pepper in [23]. A relation between the annihilation number and independence number was considered by the same author in [21].

The annihilation process is very similar to a reduction process on the degree sequence called the *Havel-Hakimi* process [16, 24]. The Havel-Hakimi algorithm gives an answer if there exists for a finite list of nonnegative integers a simple graph such that its degree sequence is exactly this list.

We call a set S of vertices an *annihilation set* if $\sum_{v \in S} d(v) \leq m(G)$, and S is an *optimal annihilation set*, if

$$|S| = a(G) \quad \text{and} \quad \max\{d(v) \mid v \in S\} \leq \min\{d(u) \mid u \in V(G) \setminus S\}.$$

The following conjecture can be found in a slightly different form in Graffiti.pc [12], and was restated in [10] as follows.

Conjecture 1 [11, 12] *If G is a connected graph with at least 2 vertices, then $\gamma_2(G) \leq a(G) + 1$.*

By definition, every graph satisfies $a(G) \geq \left\lfloor \frac{n(G)}{2} \right\rfloor$. Also, if $\delta(G) \geq 3$, it was observed in [6] that the 2-domination number is at most $\left\lfloor \frac{n(G)}{2} \right\rfloor$. Hence, if $\delta(G) \geq 3$, then $\gamma_2(G) \leq a(G)$ clearly holds, even if G is disconnected. Therefore, it is interesting to study this conjecture for graphs with small minimum degree, i.e. $\delta(G) \in \{1, 2\}$. So far, Conjecture 1 has been proved for only one further important graph class. The result was established for trees by Desormeaux, Henning, Rall, and Yeo in 2014.

Theorem 2 [11] *For a tree T the following hold.*

(a) $\gamma_2(T) \leq a^*(T)$.

(b) $\gamma_2(T) \leq a(T) + 1$.

(c) $\gamma_2(T) = a(T) + 1$ *if and only if* $T \in \mathcal{T}$.

The value $a^*(T)$ in Theorem 2 is called the *upper annihilation number* (of a tree) and denotes the largest integer k such that the sum of the first k terms of the degree sequence of T arranged in the non-decreasing order is at most $m(T) + 1$, and $\mathcal{T}$ is a family of trees which was also defined by Desormeaux et al. [11, Definition 2]. Another proof of Theorem 2(b) was given as a corollary by Lyle and Patterson [22], and in this paper we give another, much shorter proof of the same result. The reason for this is that the idea of the proof is later used for proving our main theorem.

It is interesting to note, that Conjecture 1 still holds if we replace the 2-domination number with the total domination number of a tree, and this result was proved by Desormeaux, Haynes, Henning in [10]. The result was recently extended for the family of cactus and block graphs by Bujtás and Jakovac [4].

In this paper we prove Conjecture 1 for the family of block graphs, which is one of natural generalizations of trees. Block graphs are a well-studied class with several applications, for instance [3]. The main result is the following one.

Theorem 3 *If G is a nontrivial block graph, then $\gamma_2(G) \leq a(G) + 1$.*

The paper is organized as follows. In Section 2, we establish a lemma and another, shorter proof of Theorem 2(b). In Section 3 we prove Theorem 3. In the proof we follow the same line of thought as in [4] with some key differences. In the last section we discuss the possibilities of using similar methods for proving the conjecture for cactus graphs.

2 Another, shorter proof for trees

In this section we prove a useful lemma which concentrates on strong support vertices of graph G.

Lemma 4 *Assume that G is a graph on at least four vertices and $u \in V(G)$ is a strong support vertex which is the common neighbor of pendant vertices $v_1, \ldots, v_\ell \in V(G)$, $\ell \geq 2$. If $G' = G - \{u, v_1, \ldots, v_\ell\}$ is a connected graph, then $\gamma_2(G') \leq a(G')+1$ implies $\gamma_2(G) \leq a(G)+1$.*

Proof. By our assumption $G' = G - \{u, v_1, \ldots, v_\ell\}$ is connected, and we have $m(G') = m(G) - d_G(u)$. Suppose that $\gamma_2(G') \leq a(G') + 1$. If D' is a minimum 2-dominating set of G', then $D = D' \cup \{v_1, \ldots v_\ell\}$, $\ell \geq 2$, is a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D| = |D'| + \ell = \gamma_2(G') + \ell$. Regardless of whether the neighbors of u are inside S' or not we have

$$\sum(S', G) \leq \sum(S', G') + d_G(u) - \ell \leq m(G) - \ell.$$

Let $S = S' \cup \{v_1, \ldots, v_\ell\}$. Then, we have

$$\sum(S, G) = \sum(S', G) + d(v_1) + \cdots + d(v_\ell) = \sum(S', G) + \ell \leq m(G),$$

and hence $a(G) \geq |S| = |S'| + \ell = a(G') + \ell$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + \ell \leq a(G') + \ell + 1 \leq a(G) + 1.$$

□

In the remainder of this section we give another, shorter proof of Theorem 2(b). For the purpose of the proof in Section 3 we state it again with an alternative formulation.

Theorem 5 *If T is a nontrivial tree, then $\gamma_2(T) \leq a(T) + 1$.*

Proof. We proceed by induction on the number of vertices n of T. For $n = 2$ we have $T \cong K_2$, and $\gamma_2(K_2) = 2 = a(K_2) + 1$. For the inductive hypothesis, let $n \geq 3$, and assume that for every nontrivial tree T' with less than n vertices we have $\gamma_2(T') \leq a(T') + 1$. Let T be a tree of order n. If $\mathrm{diam}(T) = 2$, then T is a star $K_{1,n-1}$, and we have $a(T) = n - 1$ and $\gamma_2(T) = n - 1$, which implies $\gamma_2(T) \leq a(T) + 1$. Hence, we may suppose that $\mathrm{diam}(T) \geq 3$.

In the proof tree T' will be formed from T by removing a set of vertices in such a way that T' will still be a tree. Throughout, S' will denote an optimal annihilation set in T'.

Let u and z be two endvertices of a longest path in T. We root tree T in u. Then both u and z must be leaves (pendant vertices). Let w be the parent of z, and v be the parent of w. Since $\mathrm{diam}(T) = d(u,z) \geq 3$, $v \neq u$. If w is a strong support vertex, then we can settle this case with Lemma 4. Thus, we may suppose that w is not a strong support vertex and has degree $d(w) = 2$. We denote with $v_1, \ldots, v_b$, $b \geq 1$, the leaf-neighbors of vertex v if it has any. For $b = 0$ this means that v has no leaf-neighbors. We also denote with $w_1, \ldots, w_c$, $c \geq 1$, the children of vertex v which are not leaves (note that we have at least one such vertex, namely w). Each of these children must be support vertices, since $d(u,z)$ is a longest distance in tree T. If any vertex of $w_1, \ldots, w_c$ is a strong support vertex, we can again settle this case with Lemma 4. Thus, we may suppose that $d(w_i) = 2$ for all $i \in \{1, \ldots, c\}$. For any $i \in \{1, \ldots, c\}$ we denote with z_i the only child of w_i (note that z is one of those children). It is also clear that $d_T(v) = b + c + 1$ because tree T is rooted in u (Figure 1).

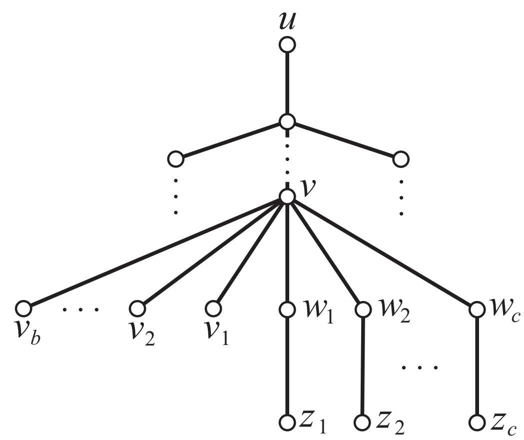

Figure 1: Tree T rooted in vertex u with all of its descendants.

Now, let $T' = T - \{v_1, \ldots, v_b, w_1, \ldots, w_c, z_1, \ldots, z_c\}$, and so $m(T') = m(T) - b - 2c$, and $d_{T'}(v) = 1$. Vertex v must belong to any minimum 2-dominating set of T', since it is a leaf. Hence, if D' is a minimum 2-dominating set of T', then $D = D' \cup \{v_1, \ldots, v_b, z_1, \ldots, z_c\}$ is a 2-dominating set of tree T, which implies $\gamma_2(T) \leq |D| = |D'| + b + c = \gamma_2(T') + b + c$. If $v \notin S'$, then $\sum(S', T) = \sum(S', T') \leq m(T) - b - 2c$. In this case let $S = S' \cup \{v_1, \ldots, v_b, z_1, \ldots, z_c\}$. Then, for $c \geq 1$,

$$\begin{aligned}\sum(S,T) &= \sum(S',T) + d(v_1) + \cdots + d(v_b) + d(z_1) + \cdots + d(z_c) \\ &= \sum(S',T) + b + c \leq m(T) - b - 2c + b + c = m(T) - c \leq m(T).\end{aligned}$$

We have $a(T) \geq |S| = |S'| + b + c = a(T') + b + c$. Applying our inductive hypothesis to T', we get $\gamma_2(T') \leq a(T') + 1$. Therefore,

$$\gamma_2(T) \leq \gamma_2(T') + b + c \leq a(T') + b + c + 1 \leq a(T) + 1.$$

But if $v \in S'$, then $\sum(S',T) = \sum(S',T') + b + c \leq m(T') + b + c = m(T) - c$. Let

$S = (S' \backslash \{v\}) \cup \{v_1, \ldots, v_b, w_1, z_1, \ldots, z_c\}$. Since $d_T(v) = b + c + 1$, we have for $c \geq 1$,

$$\begin{aligned}\sum(S,T) &= \sum(S',T) - d(v) + d(v_1) + \cdots + d(v_b) + d(w_1) + d(z_1) + \cdots + d(z_c) \\ &= \sum(S',T) - b - c - 1 + b + 2 + c = \sum(S',T) + 1 \leq m(T) - c + 1 \leq m(T),\end{aligned}$$

and hence $a(T) \geq |S| = |S'| + b + c = a(T') + b + c$. Applying our inductive hypothesis to T', we have that $\gamma_2(T') \leq a(T') + 1$. Therefore,

$$\gamma_2(T) \leq \gamma_2(T') + b + c \leq a(T') + b + c + 1 \leq a(T) + 1.$$

□

3 Block graphs

Recall that a block graph is a connected graph in which every 2-connected component (block) is a clique. Block graphs have minimum degree at least 3 if their building blocks are complete graphs K_k, $k \geq 4$. Thus, Conjecture 1 obviously holds for them. On the other hand, block graphs also contain blocks K_2 and K_3, and therefore, it clearly makes sense to study Conjecture 1 on block graphs.

If all cliques in a block graph are K_2, then it is a tree. For every $k \geq 3$ we will call the complete graph K_k a *complex clique*. If a block graph has two complex cliques K^1 and K^2, then we define

$$d\left(K^1, K^2\right) = \min\left\{d(u,v) \,|\, u \in V\left(K^1\right), v \in V\left(K^2\right)\right\}$$

where $d(u,v)$ denotes the distance between vertices u and v. Let $x_1 \in V\left(K^1\right)$ and $x_2 \in V\left(K^2\right)$ be two vertices such that $d(x_1, x_2) = d(K^1, K^2)$. Then we call x_1 and x_2 *exit-vertices* of complex cliques K^1 and K^2, respectively. A complex clique will be called an *outer complex clique* if it has at most one exit-vertex. If a block graph is not a tree, then by its definition it must contain at least one outer complex clique. Note that a block graph, which is neither a tree nor a clique, does not contain exit-vertices if and only if it contains exactly one complex clique, say K_k, $k \geq 3$. In this case, we will take an arbitrary vertex of the unique complex clique K_k whose degree is at least k for the role of the exit-vertex. In the right-hand side graph of Figure 2, we have four possibilities for the choice of that vertex (either x_1 or x_2 or x_3 or x_4).

Now, we are ready to present a proof of Theorem 3 and we recall its statement.

Theorem 3. *If G is a nontrivial block graph, then $\gamma_2(G) \leq a(G) + 1$.*

Proof. We proceed by induction on the number of vertices n of block graph G. For $n = 2$, we have $G \cong K_2$, and $\gamma_2(K_2) = 2 = a(K_2) + 1$. For the inductive hypothesis, let $n \geq 3$, and assume that for every nontrivial block graph G' with less than n vertices we have $\gamma_2(G') \leq a(G') + 1$. If G does not contain complex cliques, then it is a tree, and by Theorem 5 the result follows. Also, if G is a clique, i.e. $G \cong K_\ell$, $\ell \geq 2$, we have $\gamma_2(K_\ell) = 2 \leq a(K_\ell) + 1$. Thus, we my suppose that G is neither a tree nor a clique, but contains at least one complex

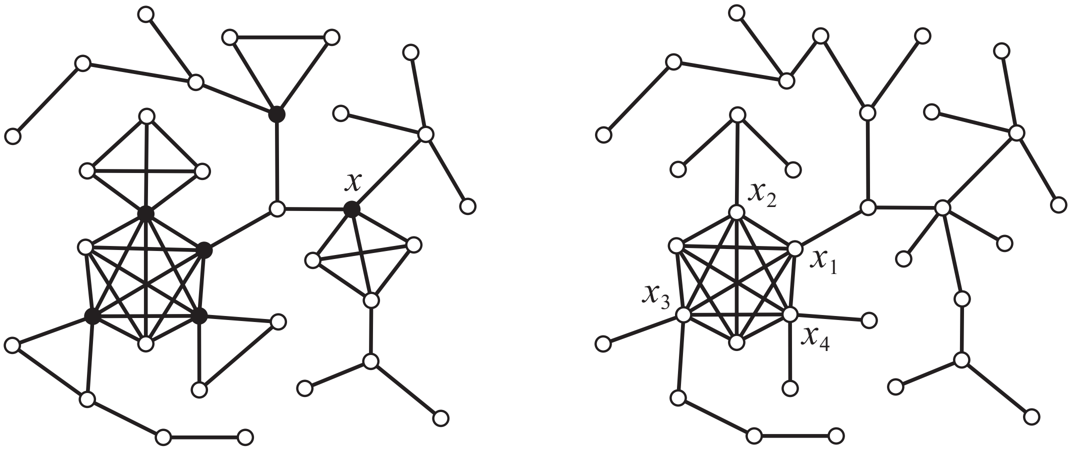

Figure 2: Two examples of block graphs. The first one has five outer complex cliques (two cliques K_4 and three cliques K_3) each having only one exit-vertex (filled with black). The second block graph has only one complex clique, and has no exit-vertices.

clique as a proper subgraph. We denote with K_k an outer complex clique of G. All outer complex cliques in the figures will be drawn with an exit-vertex x even though a unique complex clique in a block graph does not have one. As we mentioned before, in the later case, we denote with x an arbitrary vertex of clique K_k whose degree is at least k. We denote all vertices in $V(K_k)\backslash\{x\}$ with $u_1, \ldots, u_{k-1}$.

Throughout the proof, we will consider block graphs G' formed from G by removing a set of vertices in such a way that graph G' will still be a connected block graph, and S' will denote an optimal annihilation set in G'. We consider two cases.

Case 1: All vertices from $V(K_k)\backslash\{x\}$ have degree $k-1$.
We analyze two subcases with respect to k.

Case 1.1: $k \geq 4$.
Let $G' = G - \{u_3, \ldots, u_{k-1}\} - u_1u_2$, and so $m(G') = m(G) - \frac{k(k-1)}{2} + 2$. Then $d_{G'}(u_1) = d_{G'}(u_2) = 1$. Vertices u_1 and u_2 must both belong to any minimum 2-dominating set of G', since they are both pendant vertices in G'. Hence, if D' is a minimum 2-dominating set of G', then it is also a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D'| = \gamma_2(G')$. Regardless of whether x, u_1 and u_2 are inside S' or not we have

$$\begin{aligned}\sum(S', G) &\leq \sum(S', G') + 3k - 8 \leq m(G') + 3k - 8 \\ &= m(G) - \frac{k(k-1)}{2} + 2 + 3k - 8 = m(G) - \frac{k^2 - 7k + 12}{2}.\end{aligned}$$

Since $k^2 - 7k + 12 \geq 0$ for $k \geq 4$, we have $\sum(S', G) \leq m(G)$, which implies $a(G) \geq |S'| = a(G')$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') \leq a(G') + 1 \leq a(G) + 1.$$

Case 1.2: $k = 3$.

This case will be more involved than the previous one, since we only have two vertices, u_1 and u_2, to work with.

First suppose that $d_G(x) = 3$. Let $G' = G - \{u_1, u_2\}$, and so $m(G') = m(G) - 3$. Then $d_{G'}(x) = 1$. Vertex x must belong to any minimum 2-dominating set of G', since it is a pendant vertex. Hence, if D' is a minimum 2-dominating set of G', then $D = D' \cup \{u_1\}$ is a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D| = |D'| + 1 = \gamma_2(G') + 1$. If $x \notin S'$, then $\sum(S', G) = \sum(S', G') \leq m(G) - 3$. In this case let $S = S' \cup \{u_1\}$. Then

$$\sum(S, G) = \sum(S', G) + d(u_1) = \sum(S', G) + 2 \leq m(G) - 3 + 2 \leq m(G),$$

which implies $a(G) \geq |S| = |S'| + 1 = a(G') + 1$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + 1 \leq a(G') + 2 \leq a(G) + 1.$$

But if $x \in S'$, then $\sum(S', G) = \sum(S', G') + 2 \leq m(G) - 1$. Let $S = (S' \backslash \{x\}) \cup \{u_1, u_2\}$. Since $d_G(x) = 3$, it follows that

$$\begin{aligned}\sum(S, G) &= \sum(S', G) - d(x) + d(u_1) + d(u_2)\\ &= \sum(S', G) - 3 + 2 + 2 \leq m(G) - 1 - 3 + 4 = m(G),\end{aligned}$$

which implies $a(G) \geq |S| = |S'| + 1 = a(G') + 1$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + 1 \leq a(G') + 2 \leq a(G) + 1.$$

Now, let $d_G(x) = \ell \geq 4$ and denote the neighbors of x outside clique K_3 with $v_1, v_2, \ldots, v_{\ell-2}$. Since G is a block graph, the connected components of the induced graph on vertices $v_1, \ldots, v_{\ell-2}$ must be cliques. Denote these cliques with $Q_1, \ldots, Q_p$, $p \leq \ell - 2$.

If $p = 1$, we take $G' = (G - \{u_1, u_2\}) - xv_2 - \cdots - xv_{\ell-2}$, and so $m(G') = m(G) - \ell$. Clearly, G' is also connected, and $d_{G'}(x) = 1$. Vertex x must belong to any minimum 2-dominating set of G', since it is a pendant vertex. Hence, if D' is a minimum 2-dominating set of G', then $D = D' \cup \{u_1\}$ is a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D| = |D'| + 1 = \gamma_2(G') + 1$. If $x \notin S'$, then $\sum(S', G) = \sum(S', G') \leq m(G) - \ell$. In this case let $S = S' \cup \{u_1\}$. Then we have for $\ell \geq 4$,

$$\sum(S, G) = \sum(S', G) + d(u_1) = \sum(S', G) + 2 \leq m(G) - \ell + 2 \leq m(G),$$

which implies $a(G) \geq |S| = |S'| + 1 = a(G') + 1$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + 1 \leq a(G') + 2 \leq a(G) + 1.$$

But if $x \in S'$, then $\sum(S', G) = \sum(S', G') + \ell - 1 \leq m(G') + \ell - 1 = m(G) - 1$. Let $S = (S' \backslash \{x\}) \cup \{u_1, u_2\}$. Since $d_G(x) = \ell$, we have for $\ell \geq 4$,

$$\begin{aligned}\sum(S, G) &= \sum(S', G) - d(x) + d(u_1) + d(u_2)\\ &= \sum(S', G) - \ell + 2 + 2 \leq m(G) - \ell + 3 \leq m(G),\end{aligned}$$

and hence $a(G) \geq |S| = |S'| + 1 = a(G') + 1$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + 1 \leq a(G') + 2 \leq a(G) + 1.$$

Now suppose that $p \geq 2$. For every $i \in \{1, \ldots, p\}$ we choose an arbitrary vertex of clique Q_i and denote it with w_i. Note, that for every $i \in \{1, \ldots, p\}$ vertex w_i is one of the vertices $v_1, \ldots, v_{\ell-2}$. We define $G' = (G - \{x, u_1, u_2\}) + w_1w_2 + \ldots + w_1w_p$, and so $m(G') = m(G) - \ell - 1 + (p-1) = m(G) - \ell + p - 2$. Note that by adding edges $w_1w_2, \ldots, w_1w_p$, graph G' will become a connected (block) graph. Also, the degrees of vertices $w_2, \ldots, w_p$ are the same in G' and in G. Only the degree of w_1 is higher in G' than in G by exactly $p-2$. Let D' be a minimum 2-dominating set of G', and define $D = D' \cup \{x, u_1\}$ in G. Since x belongs to D, it is clear that D will be a 2-dominating set in G, because it does not matter that edges $w_1w_2, \ldots w_1w_p$ are not present in G. This implies that $\gamma_2(G) \leq |D| = |D'| + 2 = \gamma_2(G') + 2$. Regardless of whether $v_1, \ldots, v_{\ell-2}$ are inside S' or not we have

$$\begin{aligned}\sum(S', G) &\leq \sum(S', G') + (\ell - 2) - 2(p-1) \leq m(G') + (\ell - 2) - 2(p-1) \\ &= m(G) - \ell + p - 2 + (\ell - 2) - 2(p-1) = m(G) - p - 2.\end{aligned}$$

Since $p \geq 2$, we have $\sum(S', G) \leq m(G) - 4$. Let $S = S' \cup \{u_1, u_2\}$. Then, we have

$$\sum(S, G) = \sum(S', G) + d(u_1) + d(u_2) = \sum(S', G) + 2 + 2 \leq m(G),$$

and hence $a(G) \geq |S| = |S'| + 2 = a(G') + 2$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + 2 \leq a(G') + 3 \leq a(G) + 1.$$

Case 2: There exists a vertex from $V(K_k) \backslash \{x\}$ that has degree at least k.
Since $V(K_k) \backslash \{x\}$ contains vertices of degree at least k, and K_k is an outer complex clique, there are trees attached to those vertices. Suppose we root all trees in the vertices $u_1, \ldots, u_{k-1} \in V(K_k) \backslash \{x\}$ to which these trees are attached. Amongst those trees we consider the tree T with the largest height $h(T) = \max\{d(u, v) \,|\, u = V(K_k) \cap V(T), v \in V(T)\}$. Denote this maximum height with $h \geq 1$ and without loss of generality let u_1 be the vertex of $V(K_k) \backslash \{x\}$ to which tree T is attached. We consider three subcases.

Case 2.1: $h \geq 3$.
Since $h \geq 3$, there exists a leaf (pendant vertex) $z \in V(T)$ such that $d(u_1, z) = h \geq 3$. Henceforth, the proof is the same as the proof for Theorem 5. Thus, the result follows.

Case 2.2: $h = 2$.
Since $h = 2$, vertex $u_1 \in V(K_k) \backslash \{x\}$ has a path of length 2 attached to it. The only two cases which are left to consider are shown in Figure 3.

First consider the case in Figure 3(a). In this case, we suppose that vertex u_1 has at least two paths of length 2 attached to it. We denote with $v_1, \ldots, v_b$, $b \geq 2$, the children of u_1, and we may again suppose that v_i's are not strong support vertices, since otherwise Lemma 4

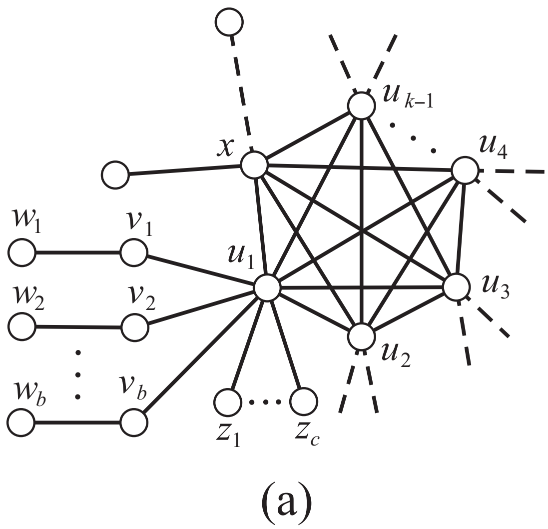

Figure 3: Cases for $h = 2$.

settles this case. For each $i \in \{1, \ldots, b\}$ let w_i be the only child of v_i. Let $z_1, \ldots, z_c$, $c \geq 0$, be any possible pendant vertices that are also attached to u_1. We consider graph $G' = G - \{v_1, \ldots, v_b, w_1, \ldots w_b\}$. Hence, $m(G') = m(G) - 2b$. If D' is a minimum 2-dominating set of G', then $D = D' \cup \{u_1, w_1, \ldots, w_b\}$ is a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D| = |D'| + b + 1 = \gamma_2(G') + b + 1$. If $u_1 \notin S'$, then $\sum(S', G) = \sum(S', G') \leq m(G) - 2b$. In this case let $S = S' \cup \{v_1, w_1, \ldots, w_b\}$. Then we have for $b \geq 2$,

$$\begin{aligned}\sum(S, G) &= \sum(S', G) + d(v_1) + d(w_1) + \cdots + d(w_b)\\ &\leq m(G) - 2b + 2 + b = m(G) - b + 2 \leq m(G),\end{aligned}$$

which implies $a(G) \geq |S| = |S'| + b + 1 = a(G') + b + 1$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + b + 1 \leq a(G') + b + 2 \leq a(G) + 1.$$

But, if $u_1 \in S$, then $\sum(S', G) = \sum(S', G') + b \leq m(G) - b$. In this case let $S = (S' \backslash \{u_1\}) \cup \{v_1, v_2, w_1, \ldots, w_b\}$. Since $d_G(u_1) = k - 1 + b + c \geq b + 2$ ($k \geq 3$ and $c \geq 0$), we have for $b \geq 2$,

$$\begin{aligned}\sum(S, G) &= \sum(S', G) - d(u_1) + d(v_1) + d(v_2) + d(w_1) + \cdots + d(w_b)\\ &\leq m(G) - b - b - 2 + 2 + 2 + b = m(G) - b + 2 \leq m(G).\end{aligned}$$

This implies $a(G) \geq |S| = |S'| + b + 1 = a(G') + b + 1$. Applying our inductive hypothesis to G', we again have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + b + 1 \leq a(G') + b + 2 \leq a(G) + 1.$$

We proceed with the case in Figure 3(b). In this case, vertex u_1 has only one path of length 2 attached to it. We denote with v_1 the only child of u_1. If vertex v_1 has more than one child, Lemma 4 again settles this case. Therefore, let w_1 be the only child of v_1. Again, let $z_1, \dots, z_c$, $c \geq 0$, be any possible pendant vertices that are also attached to u_1. Consider graph $G' = G - \{v_1, w_1, z_1, \dots, z_c\} - u_1u_2 - \cdots - u_1u_{k-1}$, and so $m(G') = m(G) - (k-2) - c - 2$, and $d_{G'}(u_1) = 1$. Note that, by removing edges $u_1u_2, \dots, u_1u_{k-1}$, graph G' is still a block graph. Vertex u_1 must belong to any minimum 2-dominating set of G', since it is a pendant vertex. Thus, if D' is a minimum 2-dominating set of G', then $D = D' \cup \{w_1, z_1, \dots, z_c\}$ is a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D| = |D'| + c + 1 = \gamma_2(G') + c + 1$. If $u_1 \notin S'$, then $\sum(S', G) = \sum(S', G') \leq m(G) - (k-2) - c - 2$. In this case let $S = S' \cup \{w_1, z_1, \dots, z_c\}$. Then, we get for $k \geq 3$,

$$\begin{aligned}\sum(S, G) &= \sum(S', G) + d(w_1) + d(z_1) + \cdots + d(z_c) \\ &\leq m(G) - (k-2) - c - 2 + 1 + c \leq m(G).\end{aligned}$$

We have $a(G) \geq |S| = |S'| + c + 1 = a(G') + c + 1$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + c + 1 \leq a(G') + c + 2 \leq a(G) + 1.$$

But, if $u_1 \in S$, then $\sum(S', G) = \sum(S', G') + (k-2) + c + 1 \leq m(G) - 1$. In this case let $S = (S' \backslash \{u_1\}) \cup \{v_1, w_1, z_1 \dots, z_c\}$. Since $d_G(u_1) = k + c$, we have for $k \geq 3$,

$$\begin{aligned}\sum(S, G) &= \sum(S', G) - d(u_1) + d(v_1) + d(w_1) + d(z_1) \dots + d(z_c) \\ &\leq m(G) - 1 - k - c + 2 + 1 + c = m(G) - k + 2 \leq m(G).\end{aligned}$$

This implies $a(G) \geq |S| = |S'| + c + 1 = a(G') + c + 1$. Applying our inductive hypothesis to G', we again have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + c + 1 \leq a(G') + c + 2 \leq a(G) + 1.$$

Case 2.3: $h = 1$.
Since $h = 1$, we have pendant vertices attached to some vertices of $V(K_k)\backslash\{x\}$. We may also suppose that vertices of $V(K_k)\backslash\{x\}$ are not strong support vertices, since otherwise we could settle this case with Lemma 4. Thus, the only two cases we need to consider are shown in Figure 4.

We start with the case in Figure 4(a). There exist a vertex different than u_1, say $u_2 \in V(K_k)\backslash\{x\}$, with degree $d(u_2) = k - 1$ (vertex u_2 has no trees attached to it). Also denote with v_1 the only child of vertex u_1, and let $G' = G - \{u_1, v_1\}$. Furthermore, we have $m(G') = m(G) - k$. Any 2-dominating set of G either contains vertex u_2 or it contains two other vertices from $V(K_k)$ that 2-dominate vertex u_2. Hence, if D' is a minimum 2-dominating set of G', then $D = D' \cup \{v_1\}$ is a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D| = |D'| + 1 = \gamma_2(G') + 1$. Regardless of whether the neighbors of u_1 are inside S' or not we have $\sum(S', G) \leq \sum(S', G') + k - 1 \leq m(G') + k - 1 = m(G) - 1$. Let $S = S' \cup \{v_1\}$. Then, we have

$$\sum(S, G) = \sum(S', G) + d(v_1) \leq m(G) - 1 + 1 = m(G),$$

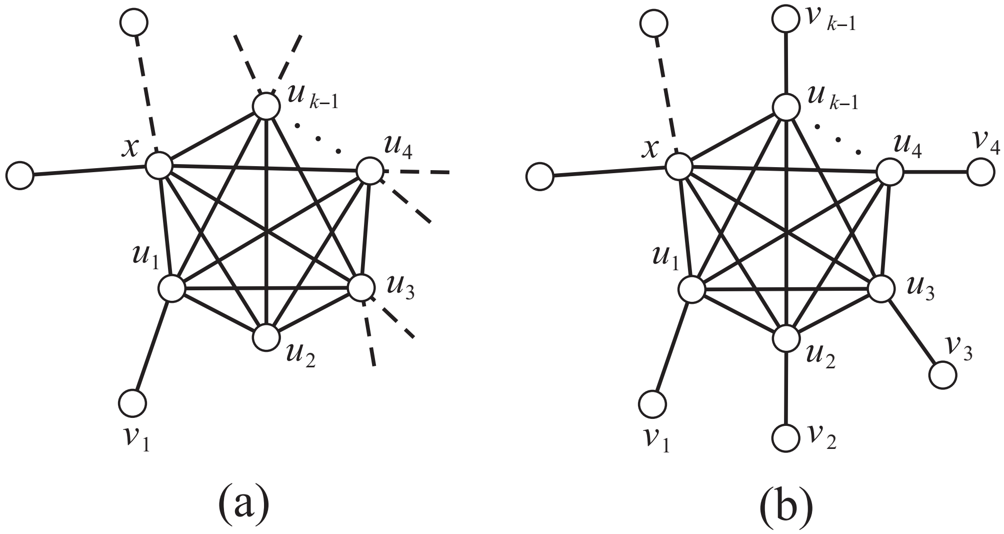

Figure 4: Cases for $h = 1$.

and hence $a(G) \geq |S| = |S'| + 1 = a(G') + 1$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + 1 \leq a(G') + 2 \leq a(G) + 1.$$

We continue with the case in Figure 4(b), and for every $i \in \{1, \ldots, k-1\}$ denote with v_i the only child of u_i. Let $G' = G - \{u_2, \ldots, u_{k-1}, v_2, \ldots, v_{k-1}\}$, and so $m(G') = m(G) - \frac{k(k-1)}{2} - k + 3$. Then $d_{G'}(u_1) = 2$ and $d_{G'}(v_1) = 1$. Vertex v_1 must clearly belong to any minimum 2-dominating set of G', since it is a pendant vertex in G'. Also, if vertex u_1 is not in D', then vertex x must be in D'. Hence, if D' is a minimum 2-dominating set of G', then $D = D' \cup \{v_2, \ldots, v_{k-1}\}$ is a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D'| = \gamma_2(G') + k - 2$. Regardless of whether x and u_1 are inside S' or not we have $\sum(S', G) \leq \sum(S', G') + \frac{k(k-1)}{2} - 1 \leq m(G) - \frac{k(k-1)}{2} - k + 3 + \frac{k(k-1)}{2} - 1 = m(G) - k + 2$. Let $S = S' \cup \{v_2, \ldots, v_{k-1}\}$. Then, we have

$$\begin{aligned}\sum(S, G) &= \sum(S', G) + d(v_2) + \cdots + d(v_{k-1}) \\ &\leq m(G) - k + 2 + k - 2 = m(G),\end{aligned}$$

and hence $a(G) \geq |S| = |S'| + k - 2 = a(G') + k - 2$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + k - 2 \leq a(G') + k - 1 \leq a(G) + 1.$$

We have considered all possible cases when deleting vertices in the block graph. Hence, Conjecture 1 is true for the family of block graphs. □

The bound in Theorem 3 is sharp since trees are also block graphs. Therefore, we may take the family of trees defined in [11, Definition 2], and used in Theorem 2, which attain the upper bound. Clearly, the bound is also sharp for the graph K_3, since $\gamma_2(K_3) = 2$ and $a(K_3) = 1$.

There are also arbitrarily large block graphs G that are not trees, but satisfy $\gamma_2(G) = a(G) + 1$. Let us define such a family of block graphs. For $k \geq 2$, we take an arbitrary tree T with the vertex set $\{u_1, \ldots, u_k\}$. For every $i \in \{1, \ldots, k\}$ we add two new vertices v_i and w_i, and make three new edges $u_i v_i$, $u_i w_i$, and $v_i w_i$. With this construction we obtain a graph G that contains exactly k triangles K_3, $k-1$ edges K_2, and has minimum degree $\delta(G) = 2$. We will call this family of graphs the *triangle-block graphs*, and denote it with $\mathcal{TB}$. An example of such a graph can be seen in Figure 5. One can easily check that $m(G) = 4k-1$ and $a(G) = 2k - 1$. For the 2-domination number of such graph, we use the fact that every triangle contains exactly two vertices of degree 2, which means that at least two vertices from every triangle must belong to any 2-dominating set. Thus, $\gamma_2(G) = 2k$, and Theorem 3 holds with equality for the family $\mathcal{TB}$.

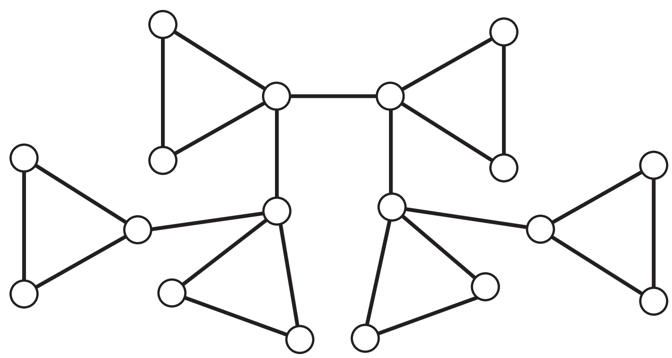

Figure 5: A graph from the family $\mathcal{TB}$.

However, graphs from the family $\mathcal{TB}$ are not the only graphs that attain the upper bound. Take for example the graph G_1 formed from K_3 with another pendant vertex attached to it (this graph is called the *paw graph*). Clearly $G_1 \notin \mathcal{TB}$. This graph has $n(G_1) = 4$ vertices and $m(G_1) = 4$ edges. It is easy to see that $\gamma_2(G_1) = 3$ and $a(G_1) = 2$. Thus, $\gamma_2(G_1) = a(G_1)+1$. Now suppose that G_2 is the graph formed from two copies of G_1 by identifying two vertices (each belonging to different copy of G_1) with degree 2. We get a graph on $n(G_2) = 7$ vertices and $m(G_2) = 8$ edges. This graph does also not belong to $\mathcal{TB}$. Again, it is easy to see that $\gamma_2(G_2) = 5$ and $a(G_2) = 4$. Thus, $\gamma_2(G_2) = a(G_2) + 1$. It almost seems that block graphs which have K_2's and K_3's for their building blocks attain the upper bound. Namely, graphs from the family $\mathcal{TB}$, and G_1 and G_2, are such graphs. But unfortunately, this is not true. Let G_3 be the graph formed from two copies of graph G_1 by identifying both pendant vertices (each belonging to different copy of G_1). This graph has $n(G_3) = 7$ vertices and $m(G_3) = 8$ edges. But, $\gamma_2(G_3) = 4$ and $a(G_2) = 4$. Thus, $\gamma_2(G_3) < a(G_3) + 1$. If we summarize everything, it is hard to find a general rule of how to form a block graph from K_2's and K_3's to attain the upper bound of Conjecture 1. Hence, it is interesting to pose the following problem.

Problem 6 *Characterize block graphs G which satisfy $\gamma_2(G) = a(G) + 1$.*

4 Concluding remarks

The conjecture $\gamma_2(G) \le a(G) + 1$ is true for all graphs G with minimum degree $\delta(G) \ge 3$. Thus the most interesting cases, for which the conjecture still remains open, are graphs with $\delta(G) = 1$ or 2. For those two cases the conjecture is known to be true for trees [11], and in this article we proved that it is also true for block graphs. Further, one could also consider other natural generalizations of trees, e.g. cactus graphs. A connected graph is called a cactus graph if its cycles are pairwise edge-disjoint (every edge belongs to at most one cycle). It is therefore natural to ask if a similar approach that we used for block graphs could be used for cactus graphs. The proof given in this paper uses induction and in each step carefully chooses an outer complex clique with only one exit-vertex, and then properly removes some vertices from this clique (together with some other vertices that lie on some trees which are connected to this clique). However, this approach does not work in the case of cactus graphs. For example, suppose we take the cactus graph in Figure 6 and chose one of its two cycles (note that both cycles have one exit-vertex x in common), and try to remove some vertices from it together with their leaf-neighbors. If we apply inductive hypothesis to the graph that remains (say that the conjecture is true for it), then it is not possible to show that also the whole graph satisfies the conjecture. Namely, the construction of a dominating set from the smaller graph to the whole graph would require much more vertices to be added to the 2-dominating set (note that all pendant vertices must belong to every 2-dominating set) than we would be able to find vertices which would contribute to the degree sum to increase the annihilation number. Thus, the 2-domination number would increase more than the annihilation number. It is therefore clear that one needs to look at the graph in Figure 6 as a whole to show that it satisfies the conjecture. Or at least, in some bigger cases, one would need to remove more vertices than just the vertices from one cycle. In some cases, where the exit-vertex x is of high degree, it would be almost impossible to control the inductive step.

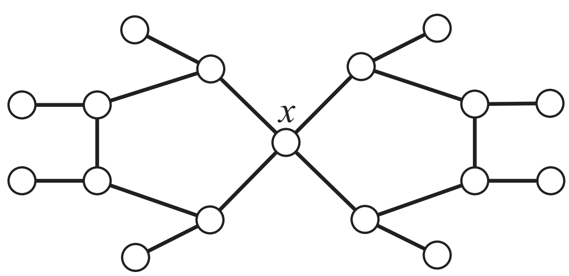

Figure 6: An example of a cactus graph

Induction is also not useful for many cactus graphs with minimum degree 2 (cactus graphs without pendant vertices). Note that the graph in Figure 5 is also such a cactus graph (even though it is also a block graph). We can even exchange any block K_3 (which is isomorphic to the cycle C_3) with any odd cycle C_{2k+1}, $k \ge 1$. Since $\gamma_2(C_{2k+1}) = k + 1$ and $a(C_{2k+1}) = k$, it is clear that with this exchange we can obtain many extremal cases in the family of cactus graphs. This fact further substantiates that proving the Conjecture 1 for cactus graphs is even harder then proving it for block graphs. However, we can at least (easily) prove it for cactus graphs which are built only of even cycles.

Proposition 7 *If G is a cactus graph such that every edge of G belongs to (exactly) one even cycle, then $\gamma_2(G) \leq a(G) + 1$.*

Proof. Let c denote the number of even cycles in G. We use induction on c. Graph G is isomorphic to a cycle C_{2k}, $k \geq 2$, for $c = 1$. Thus, $\gamma_2(C_{2k}) = k = a(C_{2k}) < a(C_{2k}) + 1$. For the inductive hypothesis let $c \geq 2$, and assume that for every cactus graph G' with less than c cycles we have $\gamma_2(G') \leq a(G') + 1$. Let S' denote an optimal annihilation set in G'. Since G is a finite graph, there exists a cycle $C_{2\ell}$, $\ell \geq 2$, in G with $V(C_{2\ell}) = \{u_1, \ldots, u_{2\ell}\}$ such that $d(u_1) \geq 4$, and $d(u_i) = 2$ for each $i \in \{2, 3, \ldots, 2\ell\}$. Let $G' = G - \{u_2, u_3, \ldots, u_{2\ell}\}$, and so $m(G') = m(G) - 2\ell$. Clearly G' satisfies the inductive hypothesis. If D' is a minimum 2-dominating set of G', then $D = D' \cup \{u_2, u_4, \ldots, u_{2\ell}\}$ is a 2-dominating set of graph G, which implies $\gamma_2(G) \leq |D| = |D'| + \ell = \gamma_2(G') + \ell$. If $u_1 \notin S'$, then $\sum(S', G) = \sum(S', G') \leq m(G') \leq m(G) - 2\ell$. In this case let $S = S' \cup \{u_2, u_4, \ldots, u_{2\ell}\}$. Then, we have

$$\sum(S, G) = \sum(S', G) + d(u_2) + d(u_4) + \ldots + d(u_{2\ell}) \leq m(G) - 2\ell + 2\ell = m(G),$$

and hence $a(G) \geq |S| = |S'| + \ell = a(G') + \ell$. Applying our inductive hypothesis to G', we have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + \ell \leq a(G') + \ell + 1 \leq a(G) + 1.$$

But if $u_1 \in S$, then $\sum(S', G) = \sum(S', G') + 2 \leq m(G') + 2 = m(G) - 2\ell + 2$. Let $S = (S' \backslash \{u_1\}) \cup \{u_2, u_4, \ldots, u_{2\ell}, u_3\}$. Since $d_G(u_1) \geq 4$, we have

$$\begin{aligned}\sum(S, G) &= \sum(S', G) - d(u_1) + d(u_2) + d(u_4) + \ldots + d(u_{2\ell}) + d(u_3) \\ &\leq \sum(S', G) - 4 + 2\ell + 2 \leq m(G) - 2\ell + 2 - 4 + 2\ell + 2 = m(G).\end{aligned}$$

This implies $a(G) \geq |S| = |S'| + \ell = a(G') + \ell$. Applying our inductive hypothesis to G', we again have that $\gamma_2(G') \leq a(G') + 1$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + \ell \leq a(G') + \ell + 1 \leq a(G) + 1.$$

□

In the end let us mention that induction was used on cactus graphs when proving the conjecture $\gamma_t(G) \leq a(G) + 1$ [4], where $\gamma_t(G)$ stands for the total domination number of a graph G. Because of the above reasoning, it is clear that another approach must be used in the case of 2-domination if one wants to prove Conjecture 1 for cactus graphs.

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