Rectangular patterns for (1,0)-relaxed $L(2,1)$-labelings of toroidal grids

1 Introduction

We use a rectangular pattern with m rows and n columns to represent a labeling of $T_{m,n}$ in a natural way. If P and Q are rectangular patterns which represent a labeling of $T_{m,n}$ and $T_{m,\ell}$, respectively, then PQ denotes the rectangular pattern with m rows and $n+\ell$ columns obtained by concatenating P and Q, such that PQ represents a labeling of $T_{m,n+\ell}$. Moreover, P^k represents a labeling of $T_{m,kn}$ made by the rectangular pattern with m rows and kn columns obtained by concatenating k copies of P.

2 Patterns for toroidal grids $T_{3,n}$

Let

$$P = \begin{bmatrix} 1 & 4 & 3 & 0 \\ 5 & 2 & 1 & 4 \\ 3 & 0 & 5 & 2 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 5 & 2 \\ 3 & 0 \\ 1 & 4 \end{bmatrix}.$$

P^k represents a (1,0)-relaxed 5-$L(2,1)$-labeling of $T_{3,4k}$, while P^kQ represents a (1,0)-relaxed 5-$L(2,1)$-labeling of $T_{3,4k+2}$.

Let

$$R = \begin{bmatrix} 6 & 0 & 5 & 3 & 1 & 4 & 5 & 1 & 3 \\ 2 & 4 & 1 & 0 & 5 & 6 & 2 & 4 & 0 \\ 1 & 3 & 6 & 4 & 2 & 0 & 3 & 6 & 5 \end{bmatrix} \text{ and}$$

$$S = \begin{bmatrix} 6 & 0 & 4 & 3 & 6 & 5 & 1 & 0 & 4 & 2 & 3 \\ 2 & 3 & 1 & 5 & 4 & 0 & 6 & 3 & 1 & 6 & 5 \\ 4 & 5 & 2 & 0 & 1 & 3 & 4 & 2 & 5 & 0 & 1 \end{bmatrix}.$$

By repeating the leftmost four columns of R and S, we obtain a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{3,4k+5}$ and $T_{3,4k+7}$, respectively, for any $k \geq 1$.

Let

$$P_3 = \begin{bmatrix} 5 & 8 & 4 \\ 3 & 6 & 2 \\ 7 & 1 & 0 \end{bmatrix}, P_5 = \begin{bmatrix} 2 & 0 & 6 & 5 & 3 \\ 4 & 5 & 1 & 7 & 6 \\ 7 & 3 & 4 & 0 & 1 \end{bmatrix}, \text{ and } P_7 = \begin{bmatrix} 2 & 0 & 6 & 1 & 4 & 3 & 5 \\ 3 & 1 & 4 & 5 & 2 & 6 & 0 \\ 6 & 5 & 2 & 3 & 0 & 1 & 4 \end{bmatrix}.$$

P_3, P_5, and P_7 represent a (1,0)-relaxed 8-$L(2,1)$-labeling of $T_{3,3}$, a (1,0)-relaxed 7-$L(2,1)$-labeling of $T_{3,5}$, and a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{3,7}$, respectively.

3 Patterns for toroidal grids $T_{4,n}$

Let

$$P = \begin{bmatrix} 4 & 0 & 2 \\ 1 & 3 & 5 \\ 0 & 2 & 4 \\ 3 & 5 & 1 \end{bmatrix}, Q_5 = \begin{bmatrix} 4 & 0 & 3 & 6 & 2 \\ 1 & 6 & 4 & 0 & 5 \\ 0 & 3 & 1 & 2 & 6 \\ 6 & 2 & 5 & 4 & 1 \end{bmatrix}, Q_7 = \begin{bmatrix} 4 & 1 & 2 & 4 & 3 & 6 & 2 \\ 0 & 3 & 5 & 6 & 1 & 0 & 5 \\ 6 & 2 & 0 & 3 & 4 & 2 & 6 \\ 3 & 5 & 6 & 1 & 0 & 5 & 1 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 7 & 2 & 3 & 5 \\ 0 & 5 & 6 & 4 \\ 3 & 1 & 2 & 7 \\ 6 & 4 & 0 & 1 \end{bmatrix}, P_5 = \begin{bmatrix} 6 & 2 & 1 & 4 & 0 \\ 1 & 5 & 3 & 6 & 2 \\ 3 & 6 & 2 & 0 & 5 \\ 4 & 0 & 5 & 3 & 1 \end{bmatrix}, \text{ and}$$

$$P_7 = \begin{bmatrix} 1 & 4 & 6 & 2 & 0 & 4 & 3 \\ 2 & 0 & 5 & 3 & 1 & 6 & 5 \\ 4 & 6 & 2 & 0 & 5 & 3 & 1 \\ 5 & 3 & 1 & 4 & 6 & 2 & 0 \end{bmatrix}.$$

For any $k \geq 1$ and $i \in \{5, 7\}$, the patterns P^k and $P^k Q_i$ represent a (1,0)-relaxed 5-$L(2,1)$-labeling of $T_{4,3k}$ and a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{4,3k+i}$, respectively. The patterns P_4, P_5 and P_7 give a (1,0)-relaxed 7-$L(2,1)$-labeling of $T_{4,4}$, a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{4,5}$ and a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{4,7}$, respectively.

4 Patterns for toroidal grids $T_{5,n}$

Let

$$P = \begin{bmatrix} 5 & 0 & 1 & 3 \\ 1 & 2 & 4 & 6 \\ 3 & 5 & 0 & 2 \\ 6 & 1 & 3 & 5 \\ 2 & 4 & 6 & 0 \end{bmatrix}, P_5 = \begin{bmatrix} 4 & 6 & 0 & 3 & 5 \\ 0 & 3 & 2 & 4 & 1 \\ 5 & 1 & 6 & 0 & 3 \\ 6 & 0 & 3 & 2 & 4 \\ 2 & 5 & 1 & 6 & 0 \end{bmatrix}, Q_5 = \begin{bmatrix} 4 & 6 & 2 & 4 & 3 \\ 0 & 5 & 3 & 0 & 6 \\ 3 & 1 & 6 & 5 & 2 \\ 6 & 4 & 0 & 3 & 4 \\ 1 & 3 & 5 & 1 & 0 \end{bmatrix},$$

$$Q_6 = \begin{bmatrix} 5 & 4 & 0 & 2 & 4 & 3 \\ 1 & 2 & 5 & 3 & 0 & 6 \\ 4 & 0 & 1 & 6 & 5 & 2 \\ 6 & 3 & 4 & 0 & 3 & 4 \\ 2 & 1 & 6 & 5 & 1 & 0 \end{bmatrix}, P_6 = \begin{bmatrix} 6 & 1 & 3 & 2 & 0 & 5 \\ 0 & 4 & 6 & 5 & 3 & 1 \\ 5 & 3 & 1 & 0 & 2 & 6 \\ 1 & 6 & 5 & 3 & 4 & 0 \\ 4 & 2 & 0 & 6 & 1 & 3 \end{bmatrix},$$

$$Q_7 = \begin{bmatrix} 5 & 1 & 3 & 6 & 2 & 0 & 4 \\ 0 & 2 & 4 & 1 & 5 & 3 & 6 \\ 3 & 5 & 0 & 3 & 6 & 4 & 0 \\ 1 & 6 & 2 & 4 & 1 & 2 & 5 \\ 2 & 4 & 5 & 0 & 3 & 6 & 1 \end{bmatrix}, \text{ and } P_7 = \begin{bmatrix} 1 & 6 & 5 & 3 & 2 & 5 & 3 \\ 0 & 4 & 2 & 0 & 6 & 4 & 2 \\ 3 & 1 & 6 & 4 & 1 & 0 & 5 \\ 2 & 5 & 0 & 2 & 5 & 3 & 6 \\ 4 & 3 & 1 & 6 & 4 & 1 & 0 \end{bmatrix}.$$

For any $k \geq 1$ and $i \in \{5, 6, 7\}$, the patterns P^k and $P^k Q_i$ constitute a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{5,4k}$ and $T_{5,4k+i}$, respectively. The patterns P_5, P_6 and P_7 give a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{5,5}$, a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{5,6}$ and a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{5,7}$, respectively.

5 Patterns for toroidal grids $T_{m,n}$

Let

$$A=\begin{bmatrix}3&6&5&1&0\\1&0&3&6&4\\5&2&4&0&3\\0&3&6&5&1\\4&1&0&3&6\end{bmatrix}, B=\begin{bmatrix}3&5&2&6&0&3&5&1&2&4&6&0\\1&0&3&1&4&2&6&4&5&1&2&4\\6&2&5&0&3&5&1&0&3&6&0&3\\0&3&6&4&1&0&4&5&1&2&5&6\\4&1&0&3&5&6&2&3&6&0&3&2\end{bmatrix},$$

$$C=\begin{bmatrix}3&6&4&2&5\\1&5&3&6&0\\4&2&0&1&3\\0&3&5&4&6\\1&6&2&0&5\\4&0&3&6&2\\3&5&1&4&0\\1&4&0&2&6\\5&2&6&1&3\\6&1&3&5&2\\0&5&2&6&4\\2&4&0&3&5\end{bmatrix}, D=\begin{bmatrix}3&6&5&1&2&4&5&0&2&4&6&5\\1&4&3&6&0&1&3&6&1&5&2&0\\5&2&0&4&3&5&0&2&4&0&3&6\\0&3&5&1&6&4&1&5&6&2&1&5\\1&6&2&0&5&2&6&3&1&4&6&3\\4&0&3&6&1&3&5&0&2&5&0&2\\3&5&1&2&4&6&1&4&6&3&4&6\\1&2&4&0&3&5&0&2&5&1&2&0\\5&0&6&5&1&2&4&6&0&4&6&3\\6&4&2&3&6&0&3&5&1&3&0&2\\0&3&5&1&2&4&6&2&4&5&1&4\\2&6&0&4&5&1&0&3&6&0&3&5\end{bmatrix}.$$

Let also $P_1 = A^pB$ and $P_2 = C^pD$ for $p \geq 1$. We now construct

$$Q=\begin{bmatrix}P_1\\ \vdots\\ P_1\\ P_2\end{bmatrix}, \text{ where } P_1 \text{ is repeated } q \text{ times and } q \geq 1.$$

Obviously, Q denotes the pattern with $5q + 12$ rows and $5p + 12$ columns.

Since the patterns A and P_1 constitute a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{5,5}$ and $T_{5,5p+12}$, respectively, while the leftmost five columns of Q constitute a (1,0)-relaxed 6-$L(2,1)$-labeling of $T_{5q+12,5}$, it follows that $T_{5p+17\alpha,5q+17\beta}$ admits a (1,0)-relaxed 6-$L(2,1)$-labeling for any $p, q, \alpha, \beta \geq 1$.